Abstract

Predicting fatigue crack growth in metals remains difficult since the available models based on the Paris law are cycle-derivative equations \( \frac{da}{dN} \), while service loads are often far from being cyclic. This imposes a cycle-reconstruction of the load sequence, which significantly modifies the load history in the signal. The main objective of this paper is therefore to propose a set of time-derivative equations for fatigue crack growth in order to avoid any cycle reconstruction. The model is based on the thermodynamics of dissipative processes. Its main originality lies in the introduction of a supplementary state variable for the crack, which allows describing continuously the state of the crack throughout any complex load sequence. The state of the crack is considered to be fully characterized at the global scale by its length \( a \), its plastic blunting \( r \), and its elastic opening. In the equations, special attention is paid to the elastic energy stored inside the crack tip plastic zone, since, in practice, residual stresses at the crack tip are known to considerably influence fatigue crack growth. The model consists finally in two laws: a crack propagation law, which is a relationship between \( \frac{dr}{dt} \) and \( \frac{da}{dt} \) and which observes the inequality stemming from the inequality of Clausius Duhem, and an elastic–plastic constitutive behaviour for the cracked structure, which provides \( \frac{dr}{dt} \) versus load and which stems from the energy balance equation. The model was implemented and tested. It successfully reproduces the main features of fatigue crack growth as reported in the literature, such as the Paris law, the stress ratio effect, and the overload retardation effect.

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1. Introduction

Predicting fatigue crack growth in metals under variable amplitude fatigue remains difficult since the available models based on the Paris law provide expressions of the cyclic crack growth rate \( da/dN = f(K_{\text{max}}, K_{\text{op}}, \text{ etc...}) \), while loads during operating conditions are often far from being cyclic. Engineering practice consists of converting original load sequences into reconstructed sequences of fatigue cycles. However, the cycle reconstruction usually significantly modifies the load history in the signal despite the fact that is has been repeatedly shown in the past 30 years that fatigue crack growth is very sensitive to load history.

The load history effect is usually taken into account in the modelling through the variations of the crack opening stress intensity factor, \( K_{\text{op}} \) [1]. The effective part of a fatigue cycle is calculated as \( \Delta K_{\text{eff}} = K_{\text{max}} - K_{\text{op}} \) and the cyclic crack growth rate is usually calculated as \( da/dN = C \Delta K_{\text{eff}} \). The determination of the evolution of \( K_{\text{op}} \) with load history can be achieved using finite element method or semi-analytical models. Since the sensitivity to load history of \( K_{\text{op}} \) includes the sensitivity of the material itself to load history [2], the finite element method is preferred over semi-analytical models because it allows the introduction of suitable material constitutive behaviours.

However, in engineering practice, finite element methods are not employed for this purpose. First of all, it is not realistic to compute millions of elastic–plastic cycles as applied to a cracked structure and even more unrealistic if a complex elastic–plastic constitutive behaviour is needed for the material. Secondly, elaborate and time-consuming methods for computing the evolution of \( K_{\text{op}} \) with load history are essentially irrelevant since a significant part of load-history is lost during the cycle-reconstruction operation.

The aim of this paper is therefore to discuss the basis and to show the versatility of a mode I fatigue crack growth model expressed as a set of time derivative equations \( da/dt \). The present model remains within the framework of fracture mechanics and is based on the thermodynamics of dissipative
processes. Its main originality lies in the introduction of a supplementary state variable for the crack which allows a continuous description of the behaviour of the crack versus time. Once this new variable is defined, the principle of virtual power, the energy balance equation, and the inequality of Clausius-Duhem are written and employed to set up the model [3].

2. Continuous characterization of the state of the crack

In practice it is common to encounter fatigue lives that stretch to tens of millions of ‘cycles’. This leads to a preference for a global approach rather than a local one. The global approach is aimed at characterizing the state of the crack through as few global variables as possible. The first state variable is the crack length \( a \). Then, for a given crack length, the displacement field around the crack tip during a load sequence in mode I is analysed in order to find how it could be characterized continuously by global variables.

While the major part of the structure remains elastic, the development of constrained plasticity in the crack tip region is the origin of dramatic history effects. Therefore, the set of global state variables should include a variable that characterizes specifically the development of crack tip plasticity.

The finite element method was employed in order to determine a suitable set of global state variables. A crack was modeled and subjected to mode I loading, under plane strain conditions. The finite element model was built in order to allow the growth of the fatigue crack. Precise details about the FEM can be found in [3]. In order to distinguish between the elastic and the plastic part of the displacement of the crack faces, the same computations were always performed, either elastically or with the aid of an elastic–plastic constitutive behaviour for the material (Fig. 1).

The ‘elastic’ displacement profile \( u_{ye}(r) \), as calculated elastically, illustrates the square root dependency on the distance to the crack tip as predicted using linear elastic fracture mechanics. The ‘elastic–plastic’ displacement profile \( u_{yep}(r) \), as calculated using an elastic–plastic constitutive behaviour for the material, is well above the ‘elastic’ one. The so-called ‘plastic’ displacement profile \( u_{yp}(r) \) was merely calculated as the difference between the ‘elastic–plastic’ and the ‘elastic’ profiles: \( u_{yp}(r) = u_{yep}(r) - u_{ye}(r) \). It corresponds also to the permanent displacement of the crack faces after unloading the cracked structure elastically. The crack tip plastic zone is always kept well below one quarter of the \( K \)-dominance area, which ensures that LEFM can be applied. Within the crack tip plastic zone, the ‘plastic’ displacement \( u_{yp}(r) \) of the crack faces varies significantly. But at the scale of the \( K \)-dominance area (\( r < d = a/10 \)) it can be approximated by a constant value, the plastic blunting \( \rho \), calculated as follows:

\[
\rho = \frac{1}{d} \int_{r=0}^{r=d} (u_{yep}(r) - u_{ye}(r))dr.
\]

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\[
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\]
Table 1
State variables and their conjugate forces

<table>
<thead>
<tr>
<th>State variables</th>
<th>Conjugate forces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic displacement field: C</td>
<td>–</td>
</tr>
<tr>
<td>Crack length: a</td>
<td>( \phi a )</td>
</tr>
<tr>
<td>Crack tip blunting: ( \rho )</td>
<td>( \phi \rho )</td>
</tr>
</tbody>
</table>

Finally, the displacement of the crack faces can be considered as the sum of an elastic displacement and of a constant permanent displacement: \( \delta u_{\text{sep}}(r) \approx \rho + C \sqrt{r} \). Here \( \rho \) is the blunting of the crack and \( C \) is the intensity of the elastic displacement field in analogy with \( K_1 \) the intensity of the elastic stress field. The mean square error associated with this assumption is also calculated as follows:

\[
\text{error} = \sqrt{\frac{1}{2} \int_0^d \left[ \delta u_{\text{sep}}(r) - \rho - \delta u_{\text{sep}}(r) \right]^2 \, dr}.
\]

In all finite element calculations, the maximum applied stress is adjusted in order to keep the mean square error of our approximation below 10%.

For a growing crack subjected to a complex load history, the displacement profile is not so simple. However, it was shown in [3] that the same approximation can still be performed using the time-derivative of the displacement profiles of the crack faces:

\[
\frac{\delta}{\delta t} \delta u_{\text{sep}}(r) \approx \frac{\delta \rho}{\delta t} + \sqrt{r} \frac{\delta C}{\delta t}.
\]

This approximation on the velocity profiles becomes irrelevant when the crack is closed.

To summarize, provided that the crack faces are not in contact, the velocity field at crack tip is assumed to be fully characterized by only three global state variables. Each global state variable is associated with a conjugate force, as shown in Table 1.

With such assumptions, the idealized geometry of the crack is as plotted in Fig. 2. According to Creager and Paris [4], as soon as \( \rho \) becomes negligible, the elastic stress field at the tip of a blunted crack is the same as that for an ideal crack except that the origin of the coordinate system is displaced by \( \rho/2 \) behind the crack tip. This consideration is helpful when the potential energy release rate with respect to a variation of \( \rho \) is calculated.

3. Energy balance equation and Clausius Duhem inequality


The problem is assumed to be quasi-static and isothermal. In such a case, the energy balance equation is as follows:

\[
P_{\text{ext}} + Q = \frac{dU}{dt} = \frac{d\psi}{dt} + T \frac{dS}{dt} = \frac{dU_s}{dt} - Q = D
\]

Where \( \int \) is the potential energy and \( D \) the dissipation. The dissipation is assumed to be inherited from four distinct dissipative mechanisms identified as \( m_1 \), \( m_2 \), \( m_3 \) and \( m_4 \) in Eq. (3). The two first mechanisms consist of heat production by the creation of new surfaces \( (m_1) \) and by plastic deformation \( (m_2) \). The two last mechanisms correspond to the storage of elastic energy. In this problem, the storage of energy is attributed to the development of an internal stress field within the crack tip plastic zone, as opposed to the development of a plastic strain gradient at the crack tip. This residual stress field (ahead and in the wake of the crack) is well known to be responsible for the history effects which are the subject of the present modeling. Therefore the storage of elastic energy is of key importance in this problem and cannot be neglected.

The rate of variation of the stored energy is related, first of all, to the rate of variation of the plastic strain gradient at the crack tip inherited from crack tip plasticity. With the above mentioned assumptions, crack tip plasticity is characterized by the rate of plastic blunting \( d\rho/dr \). Therefore the first source of evolution of the stored energy is crack tip blunting \( (m_4) \).

Moreover, if the crack grows, the creation of new free surfaces modifies the boundary conditions of the mechanical problem, which leads to a rearrangement of the internal stress field, and consequently to a variation of the stored energy \( (m_3) \).
If conjugate forces are introduced for each mechanism, it yields, per unit of thickness:

\[
D = \left( \phi_a^e \frac{da}{dt} + \phi_p^e \frac{d\rho}{dt} \right) + \left( \phi_a^c \frac{da}{dt} + \phi_p^c \frac{d\rho}{dt} \right). \tag{3}
\]

Replacing \( D \) by its expression from Eq. (2) and rearranging terms yields:

\[
-\frac{d\Pi}{dt} = (\phi_a^c + \phi_a^e) \frac{da}{dt} + (\phi_p^c + \phi_p^e) \frac{d\rho}{dt}. \tag{4}
\]

Now, the release rate of the potential energy is the sum of the contribution of the release rate by crack growth and by plastic deformation, which leads to:

\[
-\frac{d\Pi}{dt} = - \left( \frac{\partial \Pi}{\partial a} \frac{da}{dt} + \frac{\partial \Pi}{\partial \rho} \frac{d\rho}{dt} \right). \tag{5}
\]

The release rate of the potential energy can be calculated using linear elastic fracture mechanics. It is well known that in such a case, where \( G \) is the potential energy release rate per unit of thickness:

\[
\frac{\partial \Pi}{\partial a} = G \tag{6}
\]

Moreover, it was discussed before that, with the present assumptions, an increase of the crack tip radius by \( d\rho \) was equivalent to a displacement by \(-d\rho/2\) along the \(x\)-axis of the origin of the coordinate system for the elastic crack tip stress fields (Fig. 2). Therefore,

\[
\frac{\partial \Pi}{\partial \rho} \approx -\frac{1}{2} \frac{\partial \Pi}{\partial a}. \tag{7}
\]

So, Eq. (4) yields finally

\[
G \frac{da}{dt} - \frac{G}{2} \frac{d\rho}{dt} = (\phi_a^c + \phi_a^e) \frac{da}{dt} + (\phi_p^c + \phi_p^e) \frac{d\rho}{dt}. \tag{8}
\]

Since \( a \) and \( \rho \) are independent variables, therefore Eq. (8) leads to:

\[
\phi_a^c + \phi_a^e = G = \phi_a \tag{9}
\]

and

\[
\phi_p^c + \phi_p^e = -\frac{G}{2} = \phi_p^\ast. \tag{10}
\]

Eq. (10) is a yield criterion for the cracked structure, since \( \rho \) measures crack tip plasticity and since this balance equation applies only when there is dissipation. For the sake of simplicity, positive values are employed and the subscript \( p \) is let down. Therefore the yield criterion for the cracked structure is now as follows:

\[
\phi_a^c + \phi_a^e = \frac{G}{2} = \phi_a \quad \text{where} \quad \phi_a^c = -\phi_a^e.
\]

\[
\phi_p^c + \phi_p^e = -\frac{G}{2} = \phi_p^\ast.
\]

Under mode I and plane strain conditions, the expression of \( G \) is as follows:

\[
2\phi = G = \frac{(1 - \nu^2)K_i^2}{E} \tag{11}
\]

3.2. Inequality of Clausius-Duhem: crack propagation law

Using the detailed expression of the dissipation as proposed in Eq. (3), and taking into account Eqs. (9) and (10), the second law can be expressed as follows:

\[
G \frac{da}{dt} - \frac{G}{2} \frac{d\rho}{dt} \geq 0
\]

Since \( G \) is positive, Eq. (11) implies:

\[
\frac{da}{dt} - \frac{1}{2} \frac{d\rho}{dt} \geq 0 \tag{12}
\]

This inequality is a bound for the time-derivative crack propagation law to be introduced in the model.

If the material is brittle, crack growth is not accompanied by plastic blunting. Therefore Eq. (12) obviously implies that the crack growth rate should be positive: \( da/dt \geq 0 \). This is in agreement with the physics of the mechanisms and the Griffith theory. In the general case, crack propagation and plastic blunting occur simultaneously. If \( d\rho/dt \) is positive, Eq. (12) provides a minimal bound for the crack growth rate, namely, the crack growth rate should be over one half of the blunting rate. It was shown by various authors that there is a relationship between the striation spacing (related to the amplitude of crack tip blunting over a full fatigue cycle) and the crack growth rate [5–7]. Therefore, the following crack propagation law is introduced, which observes the second law (Eq. (12)) and which is consistent with previous results in the literature once integrated over a full fatigue cycle:

\[
\frac{da}{dt} = \frac{\alpha}{2} \left( \frac{d\rho}{dt} \right) \quad \text{with} \quad \alpha \geq 1 \quad \text{and} \quad \begin{cases} x \geq 0 & \langle x \rangle = x \\ x < 0 & \langle x \rangle = 0 \end{cases} \tag{13}
\]

If this crack propagation law is applied to constant amplitude fatigue cycles, it yields the well-known \( \Delta \text{CTOD} \) equation

\[
\frac{da}{dN} = \int_{a}^{a(N+1)} \frac{da}{dt} = \frac{\alpha}{2} (\rho_{\text{max}} - \rho_{\text{min}}) \approx \frac{\alpha}{2} \frac{\Delta \text{CTOD}}{2}.
\]

The material parameter \( \alpha \) is to be adjusted from experiments, such as fractographic experiments [7] or fitted in order to obtain the best agreement between the results of one fatigue crack growth experiment and the prediction of the model.
If other damage mechanisms are contributing to fatigue crack growth, (such as oxidation, for instance), their contribution to the crack extent could now be merely added to the contribution of fatigue:

\[
\frac{da}{dr} = \frac{da}{dr} \bigg|_{\text{fatigue}} + \frac{da}{dr} \bigg|_{\text{oxidation}} = \frac{\alpha}{2} \left(\frac{d\rho}{dr}\right) + \frac{da}{dr} \bigg|_{\text{oxidation}}
\]

(14)

3.3. Summary

To sum up this section in a few words, the second law provides an inequality between the plastic blunting rate and the crack growth rate. This inequality is a bound for the time-derivative propagation law as introduced in the model.

A crack propagation law was chosen, which is in agreement with the literature once integrated over a full fatigue cycle and which observes the Clausius–Duhamel inequality. This crack propagation law contains only one adjustable parameter, \( \alpha \).

The energy balance equation provides a yield criterion for the cracked structure, where the thermodynamic force \( \phi \) as opposed to the crack tip blunting rate \( (d\rho/dr) \) should be equal, when there is dissipation, to the sum of a term \( \phi^{\text{eff}} \) as opposed to the heat production by plastic deformation at crack tip and of a term \( \phi^X \) as opposed to the storage of elastic energy within the crack tip plastic zone: \( \phi = \phi^X + \phi^{\text{eff}} \).

The evolution equations of \( \phi^{\text{eff}} \) and of \( \phi^X \) with respect to an increase of the crack length and to a variation of crack tip blunting are still needed. These equations are called the constitutive equations for the cyclic plasticity of the cracked structure.

4. Constitutive equations for the cyclic plasticity of the cracked structure

4.1. Unzooming by the FEM

In the previous sections, a new state variable \( \rho \) was chosen for characterizing the state of a crack continuously and using the framework of the dissipative processes, the expression of its conjugate force \( \phi \) was determined. The expression of \( \phi \) is now known as a function of the applied remote stress and \( \rho \) is calculated as explained before using the displacement profiles of the crack faces as calculated using the FEM. It is now possible to employ the FEM in order to study the evolution of \( \phi \) versus \( \rho \) as shown in Fig. 3. It is important to underline that the only experimental input needed in such analyses is the result of a cyclic stress–strain test. Using this push–pull test, the elastic–plastic constitutive law of the material can be determined.

The finite element method is employed as a scaling technique in order to determine the elastic–plastic constitutive behaviour of the cracked structure from the knowledge of the elastic–plastic constitutive behaviour of the material.

In Fig. 3 is plotted the evolution of the crack tip blunting versus the applied remote stress. On this graph an elastic domain can be identified for the cracked structure ((3) in Fig. 3), within which there is no noticeable variation of plastic blunting at crack tip. During a fatigue cycle, this elastic domain is displaced, toward positive stresses during loading and toward negative stresses during unloading ((4) in Fig. 3). The displacement of the elastic domain of the cracked structure is inherited from the storage and release of elastic energy within the crack tip plastic zone, associated with the evolution of residual stresses during forward and reverse plasticity within the crack tip region. Therefore, the evolution of the location of the centre of the elastic domain allows determining the evolution law of the conjugate force as opposed to the storage of elastic energy \( \phi^X \).

Beside this, it appears in Fig. 3, that there are two distinct thresholds for the plasticity of the cracked structure. If a cyclically increasing load is applied on the cracked structure, there is a sudden change in the rate of variation of \( d\rho/dr \) at reloading (slopes (1) and (2) in Fig. 3).

As a matter of fact, in practice there are two crack tip plastic zones, the monotonic plastic zone, and the cyclic plastic zone. The cyclic plastic zone is embedded inside the monotonic one. Therefore, it can be assumed that there are two distinct plasticity thresholds for the cracked structure. The first one corresponds to the plasticity within the cyclic plastic zone and the second one to the extent of the monotonic plastic zone. The ‘cyclic’ threshold is reached first. A first set of evolution equations for \( \rho \) can be proposed. Then the ‘monotonic’ threshold is activated, and a second set of evolution equations is also necessary. Therefore, four internal variables are now necessary. The first one \( \phi_{\text{m}}^{\text{eff}} \) is opposed to plastic deformation within the monotonic plastic zone. The second one, \( \phi_m^X \), is opposed to the storage of elastic energy within the monotonic plastic zone. The third one, \( \phi_{\text{c}}^X \), is opposed to plastic deformation within the cyclic crack tip plastic zone. And the last one, \( \phi_{\text{c}}^{\text{eff}} \), is opposed to the storage of elastic energy within the cyclic plastic zone.
The same conjugate state variable is employed for all of them, namely $r$.

The correspondence of each internal variable with the results of the FEM is detailed in Fig. 4. $\phi_{\text{eff}}$ measures the dimension of the elastic-domain of the cyclic plastic zone, while $\phi^X$ measures the displacement of that elastic domain. Similarly $\phi_{\text{eff}}^m$ measures the dimension of the elastic domain of the monotonic plastic zone. The elastic domain of the monotonic plastic zone is limited by the contact of the crack faces. The contact point corresponds to $\phi^X$. In Fig. 4, it remains equal to zero, but various calculations were performed with a non-stationary crack subjected to complex load history in order to determine the evolution equation for $\phi^X_m$.

### 4.2. Evolution equations

Numerous analyses have been performed in order to determine a set of equation which is as simple as possible, which allows successful reproduction of the finite element results, and for which the determination of the material parameter is easy. One set of equations was proposed in [3], but since that publication we have significantly improved our equations. The new set of evolution equations for the cyclic-plasticity of the cracked structure is as follows:

- **Extension of the monotonic plastic zone:** $f_m = 0$ and $\frac{dp}{dr} > 0$

  - Crack propagation law:
    \[
    \frac{da}{dr} = \alpha \frac{dp}{dr}
    \]
  - Yield criterion ($\phi$ and $\phi_m > 0$, $\phi_{\text{xm}} < \phi$):
    \[
    f_m = [\phi - \phi_{\text{xm}}]^2 - \phi_m^2
    \]
  - Normality flow rule:
    \[
    \frac{dp}{dr} = \Lambda \frac{\partial f_m}{\partial \phi}
    \]

- **Reverse plasticity in the cyclic plastic zone:**

  - Crack propagation law:
    \[
    \frac{da}{dr} = \left( -\alpha \frac{dp}{dr} \right)
    \]
  - Yield criterion:
    \[
    f_c = [\phi - \phi_{\text{xc}}]^2 - \phi_c^2
    \]
  - Normality flow rule:
    \[
    \frac{dp}{dr} = \Lambda \frac{\partial f_c}{\partial \phi}
    \]

**Consistence:**
\[
\frac{df_m}{dr} = 0
\]

**Evolution equations**
\[
\frac{\partial f_m}{\partial p} = C_m f_m^{\text{pm}}, \quad \frac{\partial f_m}{\partial a} = K_m f_m
\]
\[
\frac{\partial f_{\text{xm}}}{\partial p} = a_{\text{xm, pos}} \sqrt{\phi_{\text{xm}}}, \quad \frac{\partial f_{\text{xm}}}{\partial a} = K_{\text{xm}} \phi_{\text{xm}} + C_{\text{xm}} f_m
\]

**Material parameters** (6 are required, as discussed below)

The first four material parameters $C_m$, $p_m$, $K_m$, $a_{\text{xm, pos}}$ are determined automatically using the method of the mean square error. In Fig. 5 is provided a comparison between the evolution of $\rho$ versus the remote stress as calculated using the FEM or using the integration of the evolution equation $\frac{df_m}{dp}$.

The material parameter $C_m$ and $p_m$ are adjusted automatically using the method of the mean square error.

**Reverse plasticity in the cyclic plastic zone:**

- **Consistence:**
  \[
  \frac{df_c}{dr} = 0
  \]

**Evolution equations**
\[
\frac{\partial f_c}{\partial p} = C_n f_c^{\text{pm}}, \quad \frac{\partial f_c}{\partial a} = K_n f_c
\]
\[
\frac{\partial f_{\text{xm}}}{\partial p} = a_{\text{xm, pos}} \sqrt{\phi_{\text{xm}}}, \quad \frac{\partial f_{\text{xm}}}{\partial a} = K_{\text{xm}} \phi_{\text{xm}} + C_{\text{xm}} f_m
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The material parameter $C_m$ and $p_m$ are adjusted automatically using the method of the mean square error.
5. Application of the model

The equations were implemented initially with an explicit integration scheme [3] and at present with the use of an implicit integration scheme.

The material parameters of the cyclic constitutive behaviour of the cracked structure are determined automatically using the finite element method. No information is required from fatigue crack growth experiments. The only experiment necessary is a cyclic push–pull test on a bar in order to determine the parameters of the cyclic constitutive behaviour of the material.

The single parameter of the crack propagation law α is here adjusted to α = 1.

There are two points that have to be checked.

First is the capability of the model to reproduce the cyclic plastic behaviour of the crack for a complex loading scheme. This consists in a validation of the elastic–plastic constitutive equations for the cracked structure.

Second, the capability of this model to reproduce the main features of fatigue crack growth needs to be evaluated. As a matter of fact, nothing guarantees that the crack growth rate in constant amplitude fatigue should obey the Paris law for instance. We need to validate that our model produces realistic behaviours.

5.1. Validation of the elastic–plastic constitutive equations for the cracked structure

First of all, the predictions of the model were compared to the results of the finite element analyses in order to validate the elastic–plastic constitutive equations for the cracked structure. Numerous calculations have been done with one example provided in Fig. 6. In this figure the evolution of crack tip blunting is plotted during a LCF + HCF fatigue test. This problem is encountered, for instance, in the design of airplane engine disks or blades [8]. HCF is known to contribute to fatigue crack growth if superimposed to a static load, as it is the case in HCF + LCF [8–10]

With the present model, as long as the amplitude of HCF cycles remains inside the elastic domain of the cyclic plastic zone, no effect of HCF can be found. Then, when cyclic plasticity occurs within the plastic zone, forward and reverse variations of ρ are found during each cycle, which contribute to fatigue crack growth as shown in Fig. 6. The role of HCF alone corresponds to the amplitude of ρ during HCF cycles (in the present case 4 nanometers per HCF cycle). However, a supplementary synergetic effect between HCF and LCF can be also observed in Fig. 6. It corresponds to a slight ratchetting effect, which is found both in the FEM and the constitutive equations approach. In the present case, the amplitude of ratchetting is around 0.4 μm within a thousand of HCF cycles.

As a conclusion of this section, the agreement between the FEM results and the simulation using the elastic–plastic constitutive equations for the cracked structure is satisfactory. Secondly, according to these calculations, there is a synergetic effect between HCF and LCF that would make the crack growth rate higher, due to history effects, than that predicted using a simple damage accumulation rule.

5.2. Constant amplitude fatigue: Paris law and stress ratio effect

In this section, the capability of a time-derivative model to reproduce the main features of constant amplitude fatigue are examined.

The stress was assumed to be sinusoidal. Various stress ratios were employed. The evolutions of the crack length were calculated using our model as a function of time and are plotted versus the number of cycles in Fig. 7(a). Then the results of the model were treated as experimental results. The cyclic crack growth rate da/dN was calculated and plotted versus the amplitude of the stress intensity factor ΔK in Fig. 7(b). It is found that the results obtained for each stress ratio obey a Paris law da/dN = CΔKm with the same
Paris exponent \( (m=5 \text{ in this case}) \). Moreover the results as calculated using the model show also a significant stress ratio effect.

The stress ratio effect in constant amplitude fatigue crack growth is usually taken into account through the variations of the crack opening level \( K_{op}/K_{max} \). The crack opening level is determined from the experiments either using load–displacement curves or from the offset between the curves (Fig. 7(b)) obtained at various stress ratios assuming that the crack growth rate obeys an intrinsic Paris law \( da/dN = C \cdot \Delta K_{eff}^m \) where \( \Delta K_{eff} = K_{max} - K_{op} \).

\[
\frac{da}{dN} = C(R) \Delta K^m = C_0 \Delta K_{eff}^m = C_0 \left( 1 - \frac{K_{op}}{K_{max}} \right)^m \left( \frac{\Delta K}{1 - R} \right)^m
\]

If we assume that there is no closure effect at \( R = 0.5 \) then \( K_{op}/K_{max} = K_{min}/K_{max} = R = 0.5 \). Then, for other stress ratios, the value of the crack opening level \( K_{op}/K_{max} \) can be deduced from the value of the coefficient \( C(R) \) of the Paris law as determined for a given stress ratio (Fig. 8).

The results as calculated using our model (Fig. 7(b)) were employed in this fashion, in order to determine the evolution of \( K_{op}/K_{max} \) as predicted using the model as a function of the stress ratio \( R \). The evolution is qualitatively in agreement with typical evolutions of \( K_{op}/K_{max} \) in the literature [11].

5.3. Overload effects

Finally, the capability of the model to reproduce typical effects observed in variable amplitude fatigue [12] was also tested. One of the most remarkable effects in variable amplitude fatigue is the overload retardation effect [13–16]. Therefore, using the model, the crack was grown under constant amplitude fatigue, with \( \sigma_{max} = 150 \text{ MPa} \) and \( \sigma_{min} = 5 \text{ MPa} \), from a length \( 2a_o = 20 \text{ mm} \) up to \( 2a_{OL} = 26.1 \text{ mm} \). At this point, the stress intensity factor was of \( K_{max} = 30 \text{ MPa m}^{1/2} \). Then, one overload is applied with a maximum stress intensity factor equal to \( K_{peak} \). The ratio \( K_{peak}/K_{max} \) was set equal to a number of possible values: 1, 1.25, 1.5, 1.75 and 2. Then, after each such overloads, the crack was grown again under constant amplitude fatigue.
with $\sigma_{\text{max}} = 150$ MPa and $\sigma_{\text{min}} = 5$ MPa. The evolution of the crack length as predicted by the model is plotted in Fig. 9. The results in Fig. 9 clearly display a delayed overload retardation effect, which is equal to $\Delta N = 175,000$ cycles for $K_{\text{peak}}/K_{\text{max}} = 2$. The overload crack retardation effect, $\Delta N$, is calculated as the difference between the number of cycles necessary to obtain a crack extent of 5 mm after the overload, with or without an overload. The evolution of $\Delta N$ with the overload ratio is plotted in the inset of Fig. 9. As expected the overload retardation effect $\Delta N$ increases with the overload ratio ($K_{\text{peak}}/K_{\text{max}}$).

Using the model, it is also found that the overload retardation effect diminishes if $K_{\text{max}}$ increases for the same value of $K_{\text{peak}}/K_{\text{max}}$. All these results are qualitatively in agreement with what is usually reported in the literature.

Finally, three global variables characterize the state of the crack at any time: the crack length $a$, the plastic blunting at crack tip $r$, and the intensity of crack opening displacement $C$.

The model is based on the thermodynamics of dissipative processes. Thermodynamics conjugate forces are introduced for each state variable. In the energy balance equation, special attention is paid to the elastic energy stored inside the crack tip plastic zone, since, in practice, residual stresses within the crack tip region are known to

6. Conclusions

In this paper, the basis of a fatigue crack growth model constituted by a set of time-derivative equations were discussed. Recognizing that service loads during operating conditions are often far from being cyclic, this new approach was developed to produce predictive capability. The classical cycle-derivative models for fatigue crack growth, based on the Paris law, imply that cycles should be extracted from the real signal. This operation usually modifies the load history in the signal. Predictive capability are reduced in the process.

The model proposed here consists of a set of time-derivative equations, which explicitly avoids cycle reconstruction. For this purpose, a supplementary state variable was introduced in order to characterize the state of a crack inside a fatigue cycle. The finite element method was employed to determine which global variable would be suitable for this purpose. The plastic blunting at crack tip $r$ was selected. It is calculated from the FEM as the average value of the permanent part of the displacement of all the nodes lying on the crack faces from $r/a = 0$ up to $r/a = 0.1$. Finally, three global variables characterize the state of the crack at any time: the crack length $a$, the plastic blunting at crack tip $r$ and the intensity of crack opening displacement $C$.

The model is based on the thermodynamics of dissipative processes. Thermodynamics conjugate forces are introduced for each state variable. In the energy balance equation, special attention is paid to the elastic energy stored inside the crack tip plastic zone, since, in practice, residual stresses within the crack tip region are known to...
considerably influence fatigue crack growth. This leads to the establishment of a yield criterion for the cracked structure, within which the conjugate force as opposed to the storage of energy inside the crack tip plastic zone appears as a kinematics hardening term.

The finite element method was employed in order to determine the empirical evolution equations for the internal variables introduced in the model, a kinematics and an effective hardening term. This leads to write a set of constitutive equations which allows calculation of the rate of crack tip blunting as a function of applied load.

In its present form the model consists in two laws:

- a cracking law, which is a relationship between \( \frac{d\rho}{dt} \) and \( \frac{da}{dt} \), which observes the inequality stemming from the second law of thermodynamics, and which contains only one adjustable material parameter \( \alpha \) to be determined from one fatigue crack growth experiment.
- an elastic–plastic constitutive behaviour for the cracked structure, which provides \( \frac{d\rho}{dt} \) versus applied load, which is inherited from the energy balance equation, and which contains a set of material parameters to be determined using the FEM and an automated identification protocol. Only one push–pull experiment is necessary to identify the constitutive behaviour of the material.

The model was integrated and tested. The parameter \( \alpha \) was set arbitrary to unity. While the model was defined independently from any crack propagation experiments, it successfully reproduces the main features of fatigue crack growth, namely the fact that the crack growth rate obeys the Paris law in constant amplitude fatigue, the existence of a stress ratio effect, the overload retardation phenomena and the sensitivity of this retardation behaviour to the overload ratio and to \( K_{\text{max}} \).

7. Prospects

Since we have developed an automated identification protocol, it is reasonable to consider providing the material parameters of the equations for the cyclic-plasticity of the cracked structure directly as a function of the material parameters, such as Young’s modulus, Poisson’s ratio, yield strength and rate and amount of kinematics hardening.

The addition of time-dependent and temperature dependent material properties may allow this theoretical framework to be expanded to the problem of non-isothermal fatigue and creep fatigue.

References