Mode I fatigue crack growth under biaxial loading

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Abstract

This paper is centred on the role of the T-stress during mode I fatigue crack growth. The effect of a T-stress is studied through its effect on plastic blunting at crack tip. As a matter of fact, fatigue crack growth is characterized by the presence of striations on the fracture surface, which implies that the crack grows by a mechanism of plastic blunting and re-sharpening (Laird C. The influence of metallurgical structure on the mechanisms of fatigue crack propagation. In: Fatigue crack propagation, STP 415. Philadelphia: ASTM; 1967. p. 131–68 [8]). In the present study, plastic blunting at crack tip is a global variable \( \rho \), which is calculated using the finite element method. \( \rho \) is defined as the average value of the permanent displacement of the crack faces over the whole \( K \)-dominance area. The presence of a T-stress modifies significantly the evolution of plastic deformation within the crack tip plastic zone as a consequence of plastic blunting at crack tip. A yield stress intensity factor \( K_Y \) is defined for the cracked structure, as the stress intensity factor for which plastic blunting at crack tip exceeds a given value. The variation of the yield stress intensity factor was studied as a function of the T-stress. It is found that the T-stress modifies significantly the yield point of the cracked structure and that the yield surface in a \( (T, K) \) plane is independent of the crack length. Finally, a yield criterion is proposed for the cracked structure. This criterion is an extent of the Von-Mises yield criterion to the problem of the cracked structure. The proposed criterion matches almost perfectly the results obtained from the FEM. The evolution of the yield surface of the cracked structure in a \( (T, K) \) plane was also studied for a few loading schemes. These results should develop a plasticity model for the cracked structure taking into account the effect of the T-stress.

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1. Introduction

Critical areas of industrial components are often subjected to complex stresses, either because multiaxial loads are applied or because residual stresses are present in the material. In plane stresses are taken into account through mode I, II and III stress intensity factors. However, out of plane stresses are usually not taken into account in fatigue crack growth criterion though there are experimental evidences of their effect in the literature.

The role of out of plane stresses is examined in the present paper through the effect of the T-stress and in mode I only. In linear elastic fracture mechanics, the T-stress is the first second order term in the asymptotic development of \( \sigma_{xx} \) at crack tip.

\[
\sigma_{xx} = \frac{K_1}{\sqrt{2\pi}r} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + T,
\]

\[
\sigma_{yy} = \frac{K_1}{\sqrt{2\pi}r} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right),
\]

\[
\sigma_{xy} = \frac{K_1}{\sqrt{2\pi}r} \cos \frac{\theta}{2} \frac{\sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{2},
\]

where \( \sigma_{xx} \), \( \sigma_{yy} \) and \( \sigma_{zz} \) are the components of the stress field on each point around the crack tip defined by \( r \) and \( \theta \) (Fig. 1). In the particular case of a through thickness crack, lying in the plane \((x, z)\), subjected to remote biaxial loads \((S_x \) and \( S_z)\), the mode I stress intensity factor and the T-stress are as follows: \( K_1 = S_x\sqrt{\pi}a \) and \( T = S_z - S_y \). Under pure remote biaxial stresses \((S_x = S_y)\), the T-stress is zero. Under uniaxial remote stresses \((S_z = 0, S_y)\), the T-stress is equal to \( T = -K_1/\sqrt{\pi}a \). Therefore, the T-stress introduces a dependency to the crack length. Let consider for instance a long crack with a length \( a_1 \) and a short crack with a length
The role of the T-stress can be studied using biaxial tests or using specimens with different evolutions of T with crack length. It was shown by various authors that the T-stress has a very significant effect on fatigue crack growth [3–6]. For instance, Tong et al. [2] show that the fatigue crack growth of a crack in a CT sample is up to ten times higher than that measured in SENT or CCT samples, with the same thickness. The authors argued that this difference is due to the T-stress, positive in CT samples and negative in CCT and SENT samples. However, the effect of the T-stress appears to depend also on the stress ratio employed in the experiments. For instance, if the stress ratio is negative, the crack growth rate is found to decrease when T increases. The same effect is also found for high positive values of R. However, when the stress ratio is zero, the effect is opposite [3–5]. Empirical models have also been proposed in the literature [5,6]. For example, Howard et al. [6] proposed an empirical equation based on their experimental results as follows: 

\[ \frac{da}{dN} = C \exp(-0.14(1 + T/S_y)) \Delta K^m, \]

where \( T \) is the biaxiality parameter (T-stress), \( S_y \) the applied stress (Fig. 1) and \( C \) and \( m \) the material parameters.

The general conclusion of this introduction is that the role of the T-stress appears to be very significant in the experiments reported in literature but that the underlying mechanisms need to be clarified. Besides, the T-stress is expected to be a suitable parameter to predict the behaviour of mechanically short cracks.

In this paper, the role of the T-stress on the elastic–plastic behaviour of the crack is examined. As a matter of fact, fatigue crack growth is closely related to the cyclic plasticity within the crack tip plastic zone. Therefore, a global variable \( \rho \) was introduced by Pommier et al. [7] in order to characterize crack tip plasticity at the global scale. This variable, the plastic blunting at crack tip \( \rho \), is calculated as the average value of the permanent displacement of the crack faces over the whole K-dominance area. The evolutions of \( \rho \) versus \( K_t \), as calculated using the finite element method, employed as an unzooming technique, allowed proposing a set of elastic–plastic constitutive equations for the cracked structure \((d\rho/dt=f(K_t) \text{ and internal variables})\). Then the fatigue crack growth rate \((da/dN)\) is expressed as a function of the crack tip blunting rate \((d\rho/dt)\), through a crack propagation law, which derives from the well-known \( \Delta \text{CTOD} \) equation \((d\rho/dN=\alpha \Delta \text{CTOD})\). The same approach is employed in the present research. The role of the T-stress is examined through its effect on the evolution of \( \rho \) at crack tip.

2. Method

The finite element method is employed in order to calculate the evolution of \( \rho \), the plastic blunting at the crack tip. According to previous analyses [7], plastic blunting at crack tip is defined as follows:

\[ \rho = \frac{1}{d} \int_{r=0}^{r=d} (u_{ep}(r) - u_y(r))dr \]  

Where \( U_{ep}(r) \), the ‘elastic’ displacement profile, is calculated in elasticity, and obeys the dependency on \( \sqrt{r} \) as predicted by the LEFM theory. The ‘elastic–plastic’ displacement profile \( u_{ep}(r) \), is calculated using any suitable elastic–plastic constitutive behaviour for the material.
The ‘plastic’ displacement profile \( u_{\text{ep}}(r) \) is calculated as the difference between the ‘elastic–plastic’ and the ‘elastic’ profiles: \( u_{\text{ep}}(r) = u_{\text{ep}}(r) - u_{\text{el}}(r) \). It corresponds to the permanent displacement of the crack faces after unloading the cracked structure.

In the model developed by Pommier et al. [7], the displacement of the crack faces is approximated by the following expression:

\[
u_{\text{ep}}(r) \approx \rho + u_{\text{el}}(r)
\]

(2)

The mean square error associated with this assumption is also calculated as follows:

\[
\text{error} = \frac{1}{d} \int_{r_0}^{r_0+d} |u_{\text{ep}}(r) - u_{\text{el}}(r)|^2 \, dr
\]

(3)

In finite element calculations, the maximum applied stress is adjusted in order to keep the mean square error of this approximation below 10%. For this purpose, the crack tip plastic zone is always kept well below one quarter of the \( K \)-dominance area, which ensures that LEFM can be applied. In such a case, plasticity is constrained within the plastic zone and at the scale of the \( K \)-dominance area the approximation proposed in Eq. (2) is reasonable. The dimension of the \( K \)-dominance area \( d \) should be typically below \( a/10 \). Besides, the mesh-refinement in this area should be such as that forward and reverse plasticity within the crack tip plastic zone could be properly captured using the finite element method. This imposes to choose a mesh size of about \( r_p/20 \), where \( r_p \) is the dimension of the cyclic plastic zone. Therefore, the mesh size within the \( K \)-dominance area is equal to \( r_p/20 \), where \( r_p = d/4 \) and \( d = a/10 \).

The finite element calculation of \( \rho \) and of the associated error was fully automated. Once the mesh is defined, each loading step is calculated in elastic–plastic conditions, then for each increment the displacement profiles are extracted. Plastic blunting \( \rho \) is calculated using Eq. (1) and the associated error is calculated using Eq. (3). The evolution of the calculated plastic blunting at crack tip can be plotted versus the applied stress intensity factor. One example is given in Fig. 2. In this figure, the error associated with the calculations of \( \rho \) was added. An error bar was also added which characterizes the relative error, over a distance \( d \) ahead of the crack tip, between the exact solution and the asymptotic development for stresses along the crack plane. This relative error was then applied to \( K_I \), which is defined using the asymptotic development. It is interesting to note that the error associated with the approximation proposed in Eq. (2) is comparable to that inherited from the asymptotic development of stresses at crack tip, which allowed defining the stress intensity factor.

It was discussed in the introduction that the paper was centred on the role of the \( T \)-stress, in particular because the \( T \)-stress could be a good candidate for modelling the behaviour of mechanically short cracks. This implies that for similar stress intensity factor the crack growth rates of short and long cracks may be different, but that if both \( \Delta K \) and \( T \) are the same for a long crack and a mechanically short crack, their crack growth rate should be the same. In the model developed in [7] the crack growth rate is related to the crack tip blunting rate. Therefore, it is important to check that the calculated evolutions of \( K \) versus \( \rho \) for given value of \( T \), are independent of the crack length. This was performed for various values of \( T \) and one example is plotted in Fig. 3. In this case \( T = 0 \), and it is obvious that the evolutions of \( K_I \) versus \( \rho \) are not dependent on the crack length. The evolution of \( K_I \) versus \( \rho \) can be considered as intrinsic.

Then a second treatment of the finite element results was proposed. Since \( \rho \) characterizes the ‘plastic’ displacement of the crack faces, a yield criterion \( (\Delta \rho > \rho_{th}) \) was introduced for the cracked structure. This yield criterion is defined as a critical value \( \rho_{th} \) of the amplitude of crack tip blunting \( \Delta \rho \), over which the crack is supposed to have undergone plastic deformation. In Fig. 2, the yield criterion \( \rho_{th} \) was chosen arbitrarily to be equal to 0.1 \( \mu m \). Then the evolution of the stress intensity factor as calculated using the finite element method allows defining a yield point for the cracked structure \( K_Y \). For cyclic loading (which is the usual case in fatigue) an elastic domain is defined for the cracked structure which dimension is \( (K_Y - K_{Y'}) \) and which centre is \( (K_Y - K_{Y'})/2 \).

![Fig. 2. Evolution of plastic blunting at crack tip, as defined in Fig. 1, during the loading and unloading of the cracked structure. Definition of the yield criterion \( \Delta \rho \) and of the initial yield point \( K_{Y'} \) and cyclic elastic domain \( (K_Y \) and \( K_{Y'}) \). 2D, plane strain, CCT specimen. Material: \( E = 200 \) GPa, \( \nu = 0.3 \). Re = 100 MPa, linear kinematics hardening \( H = 100 \) MPa.](image1)

![Fig. 3. Comparison of the results obtained for various cracks lengths and the same value of \( T \). Material: \( E = 200 \) GPa, \( \nu = 0.3 \). Re = 100 MPa, linear kinematics hardening \( H = 100 \) MPa.](image2)
The calculation of $K_{1Y}, K_{3Y}^+ \text{ and } K_{3Y}^-$ was also automated which allows plotting their evolutions when the $T$-stress is varied.

Of course the calculated value of $K_{1Y}$ is closely related to the choice of $\rho_\text{th}$. It is obvious in Fig. 2, that very small values of $\rho_\text{th}$ could arbitrarily be chosen as a yield criterion. However, though very small plastic blunting can be calculated using the finite element method, they may not be reasonable from a physical point of view. However, a physical sense can be given to $\rho_\text{th}$ if $\rho_\text{th}$ is defined using the threshold stress intensity factor of the material $\Delta K_\text{th}$.

As a matter of fact, the crack propagation law proposed by Pommier et al. [7] is as follows:

$$\frac{da}{dt} = \frac{\alpha}{2} \left( \frac{d\rho}{dt} \right)^\alpha$$

with $\alpha \geq 1$ and $\begin{cases} x \geq 0 & \langle x \rangle = x \\ x < 0 & \langle x \rangle = 0 \end{cases}$

This crack propagation law, once integrated over a full fatigue cycle, is consistent with the $\Delta$CTOD equation.

With such a definition, if the amplitude of plastic blunting at crack tip is zero (below the yield criterion), the crack is not supposed to grow. As a consequence, if the threshold stress intensity factor $\Delta K_\text{th}$ is known for the material, a suitable yield criterion $\rho_\text{th}$ should be such as that $K_{1Y} = \Delta K_\text{th}$. This provides at least a reasonable value for $\rho_\text{th}$ instead of a pure arbitrary value.

Once $\rho_\text{th}$ is chosen the same criterion is kept for all loading conditions. Then numerous finite element calculations have been conducted under biaxial remote stresses ($S_x$ and $S_y$) (Fig. 1) in order to determine the evolution of $K_{1Y}$ with the $T$-stress, $T = S_x - S_y$.

3. Results and yield criterion for the cracked structure

The evolution of $K_{1Y}$, for a given value of $\rho_\text{th}$, was plotted in Fig. 4 versus the $T$-stress as applied during the finite element calculations. The $T$-stress is found to modify very significantly the yield point of the cracked structure. The yield point goes through a maximum around $T = 0$ and then decreases when $T$ increases of decreases. The variations of the yield point with the $T$-stress are not negligible. For instance the size of the elastic domain of the cracked structure is equal to 8.5 Mpa m$^{1/2}$ for $T = 0$, and equal to 6.5 Mpa m$^{1/2}$ for $T = -50$ Mpa = $-R_c/2$.

Now the problem is to find a mechanism at the origin of such an evolution of the yield point of the cracked structure with the $T$-stress. Crack tip plasticity is assumed to obey the Von-Mises yield criterion. Besides it is considered that below the yield point of the cracked structure the material behaviour remains elastic. Therefore, the elastic stress fields at crack tip can be employed in order to calculate an extent of the Von-Mises yield criterion to the problem of the cracked structure.

Recall, the Von-Mises yield criterion is based on the elastic shear energy inside a given volume of material. A volume of material subjected to a multiaxial but homogeneous stress state is supposed to yield for the same density of elastic shear energy as a sample subjected to a uniaxial tensile test.

This leads us to propose an extension of the Von-Mises yield criterion to the problem of the cracked structure. The displacement field of the LEFM in the presence of a $T$-stress is found to modify very much the density of elastic shear energy.

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energy of a cracked structure subjected to $K_1 = 0$ and to $T \neq 0$ is equivalent to the elastic shear energy of an uncracked specimen under a uniform stress $S_v$. Its yield point is known and can be calculated using the Von-Mises yield criterion.

Taking into account the plane strain conditions, it comes that the yield point when $K_1 = 0$ is reached for:

$$T_s = \frac{R_c}{\sqrt{1 - \nu + \nu^2}}$$

After simplifications the yield criterion of the cracked structure can be alternatively written as follows:

$$\left( \frac{K_1}{K_v} \right)^2 + f_p \left( \frac{K_1}{K_v} \right) \left( \frac{T}{T_v} \right) + \left( \frac{T}{T_v} \right)^2 = 1$$

where: $T_v = R_c/\sqrt{1 - \nu + \nu^2}$, $K_v = 2\sqrt{2\pi}\sigma R_c/\sqrt{7 - 16\nu + 16\nu^2}$. Here: $f_p = (32(1 - 16\nu + 10\nu^2)/15\pi(1 - \nu + \nu^2(7 - 16\nu + 16\nu^2))^2) = 0.189 - 2.10\nu$

In Fig. 4, the yield point of the cracked structure ($K_v$) as calculated using the present criterion was compared to the yield point as calculated using the finite element method (with $\rho_{th} = 0.1 \mu$m). It can be seen in Fig. 4 that this yield criterion for the cracked structure matches almost perfectly the results stemmed from the finite element method. Therefore, according to the finite element results and to a simple plasticity theory, the $T$-stress is found to have a significant impact on the development of crack tip plasticity.

4. Evolution of the yield surface of the cracked structure

The finite element method was also employed in order to study the evolution of the elastic domain of the cracked structure ($K_{IV}, K_{II}$) after various loadings histories. It appears that the elastic domain is not significantly distorted but displaced in a $(T, K_1)$ plane (see for instance the displacement of the yield surface after a complex non-proportional loading path Fig. 5). In the future, it is expected to associate a kinematics hardening internal variable to the position of the centre of the yield surface of the cracked structure. But this should be done in the framework of the thermodynamics of dissipative processes which is not yet the case.

5. Conclusions and prospects

From the definition of crack tip blunting at crack tip, an elastic domain can be defined for the cracked structure, as the domain in a $(T, K_1)$ plane within which the variation of crack tip blunting is below a critical value ($\rho_{th} = 0.1 \mu$m, for instance). It is shown that the presence of a $T$-stress modifies significantly the yield point of the cracked structure. A criterion is proposed for the yield point of the cracked structure, which is an extension of the Von-Mises yield criterion to the problem of the cracked structure. It is found to match almost perfectly the results from the FEM. Finally, the evolution of the elastic domain of the cracked structure in a $(T, K_1)$ plane is studied for complex loadings scheme. It appears that it obeys a simple displacement rule.

In the future, the constitutive model developed by Pommier and Risbet [7] within the framework of dissipative processes should be extended to biaxial remote loading conditions. This research shows that the Von-Mises plasticity theory is suitable to explain the role of the $T$-stress on the development of crack tip plasticity. Since the fatigue crack growth rate is closely related to crack tip plasticity, this could well explain the effect of biaxial loading conditions as reported in the literature. Our first prospect is to perform fatigue crack growth experiments under biaxial conditions and to analyse the effect of the $T$-stress. Then in order to compare predictions and experiments, it is necessary to build a complete theory for the plasticity of the cracked structure for complex loading conditions, as it was done for pure mode I loading conditions ($T = 0$).

References