Non-intrusive coupled global/local analysis of localized plasticity problems

Lionel GENDRE*, Olivier ALLIX1, Pierre GOSSELET1, François COMTE2

1 LMT-Cachan (ENS Cachan/CNRS/UPMC/PRES UniverSud Paris)
61, av. du Président Wilson – F-94230 Cachan – FRANCE
*gendre@lmt.ens-cachan.fr
2 Snecma Villaroche – Rond-point René Ravaud – Réau – F-77550 Moissy-Cramayel – FRANCE

Abstract
This paper presents a computational strategy to solve structural problems featuring nonlinear phenomena that occur within a small area of interest, while the rest of the structure retains a linear elastic behavior. Two finite element models are involved: a global linear model of the whole structure, and a local nonlinear “submodel” meant to replace the global model in the area of interest. An iterative exchange technique is then used to perform this substitution in an exact but non-intrusive way, which means the model data is never modified and the computations can be carried out with standard finite element models and codes. Several ways of exchanging data between the models are investigated and convergence rates are illustrated on simple academic cases.

1. Introduction
In the aircraft industry, it is a common task to perform a finite element analysis on a complex structure that mostly evolves in a linear elastic way, but exhibits confined plasticity (or other nonlinear phenomena) in a small critical region. Such an analysis is usually carried out using a one-way submodeling technique, consisting in a global linear computation followed by local nonlinear “zooms” centered on critical zones and driven by the global displacements (as presented by Kelley in [6]) or stresses (Jara-Almonte et al., [5]). Although numerically efficient, those methods have a strong limitation: since they ignore the global influence of local plasticity, they cannot assess phenomena such as large-scale stress redistributions, and tend to introduce uncontrolled errors due to the inaccuracy of local boundary conditions.

On the other hand, several nonlinear multiscale strategies have been recently designed, such as the micro/macro approach [4,7] or the works of Cresta et al. [2]. Those methods provide robust and efficient ways to solve large nonlinear problems, but they suffer from a drawback: they are intrusive with respect to traditional finite element (FE) formulations. In other words, they cannot be carried out using commercial FE codes and industrial model data files, and their implementation into an industrial environment would be long and difficult.

This paper presents a non-intrusive and exact analysis strategy for finite element problems with localized plasticity. Much like submodeling techniques, it involves two FE models and can be carried out using standard FE codes; however, it is built around an iterative exchange procedure designed to converge to the exact FE elastic-plastic solution. A brief description of the method and some results are proposed here.

2. Essential ideas
2.1 The elastic-plastic reference problem
Let us consider an elastic-plastic problem, where all plasticity is supposed to be confined within a small region $\Omega^p$ of the structure, called the domain of interest (dark area on Figure 1). In the remaining region $\Omega^c$, called the complement domain, we assume the constitutive law is purely linear elastic. Moreover, small perturbation theory is used; therefore the only source of nonlinearity is the plasticity in $\Omega^p$. 
2.2 Global and local finite element models

As explained before, we intend to solve the elastic-plastic problem with two FE models. The first one, known as the global model, describes the whole structure and is purely linear elastic. We assume it to represent the complement domain’s mechanical behavior accurately, however there can be several inaccuracies in the domain of interest: plasticity may occur, the FE mesh may be too coarse with respect to the expected gradients, and some small geometric details such as holes or cracks may be absent from the mesh.

Therefore, a local model is used to provide an accurate representation of the area of interest, with a “correct” mesh and an elastic-plastic constitutive law. The two meshes are assumed to match on Γ, the boundary between the two domains, but can be completely different inside Ω as shown on Figure 1.

Finally, the reference mesh is the mesh one would obtain by substituting the local mesh into the global mesh, and the reference solution is the FE solution to the elastic-plastic problem discretized onto the reference mesh.

![Figure 1: Global (left), local (center) and reference (right) FE models. The dark area is elastic-plastic.](image)

2.3 Solution substitution

In many global/local approaches, the solution is searched as the sum of the global solution and a local enrichment term (as in the works of Ben Dhia [1], Duester et al. [3] or Mao & Sun [8]). This is not the case here, and we define the global-local solution by substituting the local FE solution (with superscript L) into the global solution (G). Denoting displacements by u and stresses by σ, this is written:

\[
s(\bar{x}) = (u(\bar{x}), \sigma(\bar{x})) = \begin{cases} 
s^L(\bar{x}) & \text{if } \bar{x} \in \Omega^L \\
s^G(\bar{x}) & \text{if } \bar{x} \in \Omega^C. \end{cases}
\] (1)

There are two advantages in doing this. First, the two models can be defined quite independently: there is no compatibility constraint inside Ω’. In other words, deep geometry and/or mesh modifications can be introduced via the local model, and their exact influence on the whole solution can be assessed without changing anything in the global model, which could be helpful during design processes. Second, this form enables to handle local nonlinearity at the level where it actually occurs, i.e. within the local model; Cresta’s works [2] suggest that this idea can save computational costs compared to expensive global iterations.

2.4 Iterative exchange technique

Finally, an iterative exchange technique is used to link both models and to have the global-local solution (defined by (1)) converge to the reference solution (defined in Section 2.2). The method is initialized by a global analysis that can be used to estimate the location of plastic areas. Then each iteration goes as follows:

1. Local analysis: a nonlinear analysis is performed on the local model, with part of the previous global solution specified as a boundary condition on Γ (displacement, traction or mixed).
2. Residual computation: the residual, a load vector measuring the discontinuity of the global-local solution across Γ, is formed. If its magnitude is “too large”, we then proceed to the
3. Global correction: the residual is injected as an additional load into the global problem, in order to have the global model deform as if local details were present; after updating the global solution, the process is repeated from Step 1 until the residual is small enough.

3. Discussion and results

According to the type of boundary condition chosen for the local analysis, the implementation and convergence rate of the method can be very different. Two possible choices are discussed below.
3.1 Displacement-based zooming

Displacement-based zooming is obtained when the global displacement field is specified as a boundary condition on $\Gamma$ during the local analysis. Assuming this choice, once Step 1 is complete, the global-local solution (defined by (1)) verifies every equation of the reference problem, except traction equilibrium on $\Gamma$. Therefore the residual (Step 2) is defined as the virtual work of global-local traction discontinuity across $\Gamma$, using the vector of global FE basis functions $\mathbf{N}_G$ as a set of test functions:

$$r^G = -\int_{\Gamma} (\mathbf{\sigma}_G^L \mathbf{n}_G^L + \mathbf{\sigma}_G^G \mathbf{n}_G^G) \cdot \mathbf{N}_G^G d\Gamma$$

(2)

To avoid stress interpolation errors, this vector is directly assembled from nodal reaction forces on $\Gamma$ (conjugate to the prescribed displacements). Then global correction (Step 3) can be simply performed by solving an additional global problem loaded by the residual alone, and then updating the global solution:

$$\mathbf{u}_G^G = \mathbf{u}_G^G + \mathbf{[K]}_G^{-1} r^G$$

(3)

where $\mathbf{K}_G$ is the global stiffness matrix or operator. This approach is similar to the “Iterative Global-Local” technique proposed by Whitcomb [9] and is basically a modified Newton method formulated on surface traction equilibrium. Therefore, it tends to converge slowly when plasticity occurs because $\mathbf{K}_G$ is then too stiff. This can be improved by replacing $\mathbf{K}_G$ with a tangent operator $\mathbf{K}_G$ obtained from tangent moduli:

$$\mathbf{u}_G^G = \mathbf{u}_G^G + \mathbf{[K]}_G^{-1} r^G$$

(4)

To avoid modifying the global model, $\mathbf{K}_G$ is never assembled. Instead, since $\mathbf{K}_G$ and $\mathbf{K}_G$ differ only by a few terms (actually by a small rank matrix), (4) is addressed by an iterative Krylov algorithm that uses $\mathbf{[K]}_G^{-1}$ as a preconditioner by sending load vectors to the global problem. This is demonstrated on Figure 2, where the convergence rates of three different operators are compared on a simple 1D test case – the initial global operator, a “true” tangent operator which can be proved to be optimal (i.e. fastest possible), and an approximate tangent operator. A significant improvement in convergence rates can be noticed (4 iterations with the optimal operator versus 30 with the basic method, to get a $10^{-4}$ residual error), although the computational cost of the extra calls must also be assessed.

![Figure 2: 1D test case (left) and convergence rates for the displacement-based zooming (right), using three different operators for global correction](image)

3.2 Mixed zooming

Even with an optimal global correction step, it takes several iterations to link the two models properly: as shown in Cresta’s works [2], a displacement boundary condition isn’t the most relevant choice from a mechanical point of view. To achieve a better convergence rate, one can use a mixed Robin boundary condition on $\Gamma$:

$$(\mathbf{\sigma}_G^L \mathbf{n}_G^L + \mathbf{\sigma}_G^G \mathbf{n}_G^G) = \mathbf{K}_G^{-1} (\mathbf{u}_G^G - \mathbf{u}_G^L)$$

(5)
where $\mathbf{K}^{G\rightarrow L}$, the FE discretization of $\Delta^{G\rightarrow L}$, is a positive matrix defining an additional stiffness on $\Gamma$. This kind of condition can be very accurate if $\mathbf{K}^{G\rightarrow L}$ is a good representation of the stiffness of the complement domain $\Omega^C$; in particular, one can prove that if $\mathbf{K}^{G\rightarrow L}$ is the exact Schur complement of $\Omega^C$, then the local solution equals the reference solution after the first local analysis, no matter how inaccurate the global solution is. This may be particularly interesting for applications where the only quantities of concern are in the local model.

With this method, neither the displacements nor the tractions are continuous across $\Gamma$ at the end of Step 1. Therefore the residual must measure both discontinuities, and contains an additional term with respect to (2):

$$r^G = \int_\Gamma \left[ -\sigma^G \varepsilon^C + \sigma^L \varepsilon^L \right] + (\mathbf{K}^{L\rightarrow G})^T (\mathbf{u}^L - \mathbf{u}^G) \cdot \mathbf{N}^G d\Gamma$$

For $\mathbf{K}^{L\rightarrow G}$, the FE discretization of $\Delta^{L\rightarrow G}$, is another parameter similar to $\mathbf{K}^{G\rightarrow L}$. Global correction is then performed as previously, by applying the residual as an additional global load. It was proven that if $\mathbf{K}^{L\rightarrow G}$ is the exact Schur complement of the local model, then the optimal global correction operators are the same as those defined for the displacement-based zoom. Thus, if $\mathbf{K}^{L\rightarrow G}$, $\mathbf{K}^{G\rightarrow L}$ and $\mathbf{K}^C$ all match their optimal values, then the whole global-local solution reaches the reference solution in just one iteration.

Of course, those Schur complements cannot be computed exactly at realistic costs for large industrial models. Therefore we have tested algebraic approximations to them; when using only the matrix blocks relative to the boundary ($\Gamma$) degrees of freedom, it only took 3 global computations to attain complete convergence for the 1D test case on Figure 2, and only 2 iterations to get an accurate local solution (with a relative error near $10^{-4}$).

### 3.3 More results

As of March 2008, the strategy is being implemented with Abaqus/Standard as a master script submitting and post-processing analysis jobs. Results on 2D and/or 3D test cases will be presented at the conference.

### Acknowledgment

This work is part of the MAIA-MM1 research program and is supported by Snecma.

### References


