Damage and fatigue

Continuum damage mechanics modeling for fatigue of materials and structures

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ABSTRACT. Application of damage mechanics to fatigue is addressed in the present paper. The ability of Lemaitre’s damage law to describe low and high cycle fatigue failure of metals is illustrated. Its generalization into an evolution law modeling a damage rate governed by the main dissipative mechanics applies to concrete, elastomers, rocks and to probably other materials. The importance of rate written laws is emphasized.

RÉSUMÉ. On décrit dans le présent article comment la mécanique de l’endommagement permet de modéliser la fatigue des matériaux. La loi de Lemaitre permet de rendre compte de la fatigue à faibles et à grands nombres de cycles des métaux. Sa généralisation en une loi d’évolution gouvernée par le mécanisme dissipatif principal permet de l’appliquer aux bétons, élastomères, roches et à probablement d’autres matériaux. L’importance d’une écriture en vitesse est soulignée.

KEYWORDS: damage, fatigue, metals, concrete, elastomers, rocks.

MOTS-CLÉS : endommagement, fatigue, métaux, béton, élastomères, roche
1. Introduction


– ensuring the transition from continuous damage to discrete cracking: determination of the size and orientation of the mesocrack initiated (Mazars et al., 1996, Lemaitre et al., 2005), discontinuity modeling, XFEM (Melenk et al., 1996, Moës et al., 1999, Jirasek, 2002a, Jirasek, 2002b),

– allowing the computational coupling with multi-scale and multi-timescale physics, such as for thermo-chemo-hydro-mechanics problems,

– making the computations robust and fast, such as by use of extensive parallelism.

Point probably less known, damage mechanics is also powerful for fatigue, even if this topic is usually addressed with specific engineering rules modeling directly the Wöhler (or Manson-Coffin) curves of materials (Manson et al., 1964, Aas-Jackobsen et al., 1973, Tepfers et al., 1979, Ramakrishnan et al., 1992). A fatigue law for concrete can be a straight line in the periodic maximum applied stress $\sigma_{\text{Max}}$ vs the logarithm of the number of cycles to rupture $\log N_R$ diagram, generally parametrized by the stress ratio $R_\sigma = \sigma_{\text{min}}/\sigma_{\text{Max}}$ (with $\sigma_{\text{min}}$ the minimum applied stress). Such a reference curve is 1D so how to apply it to 3D states of varying amplitude stresses? What is then a stress amplitude? How to define a stress ratio? Many difficulties arise today due to the will of making Finite Element computations for which the state of stress is far to remain uniaxial and the loading type is far to remain stress controlled, even proportional and/or cyclic.

The present paper focus on the advantages of considering CDM framework for fatigue, more precisely of considering Lemaitre’s damage law and extending it to non metallic materials as concrete, elastomers, rocks...
2. Amplitude damage laws

A first, natural, kind of damage model for fatigue is simply written by relating the damage, denoted $D$, to the number of cycles $N$. Setting for an experiment $i$ leading to a number of cycles to rupture $N_{R_i}$ for a stress amplitude $\Delta\sigma_i$,

$$D = \frac{N}{N_{R_i}}$$ \[1\]

allows to recover Miner’s linear accumulation rule for multi-level loadings made of blocks, each made of $n_i$ cycles of constant amplitude,

$$\sum_i D_i = 1 \quad \rightarrow \quad \sum_i \frac{n_i}{N_{R_i}} = 1$$ \[2\]

This formulation also defines the damage increment per cycle $\frac{\delta D}{\delta N}$ as $1/N_{R_i}$, damage increment which can more generally be set as a function of the current damage $D$, of the stress amplitude and of the stress ratio $R_\sigma$ (Lemaitre et al., 1984, Hua et al., 1984, Chaboche et al., 1988),

$$\frac{\delta D}{\delta N} = g(D) G_\sigma(\Delta\sigma, R_\sigma)$$ \[3\]

The $R_\sigma$ dependency models the effect of different mean stresses on Wöhler curves.

People initially thought that introducing a $D$ dependency as in Equation \[3\] combined with the definition of a critical damage $D_c < 1$ (such as $N_R = N(D_c)$) would recover a nonlinear (observed) damage accumulation ($\sum \frac{n_i}{N_{R_i}} \neq 1$). This is not the case as one has for a constant amplitude loading:

$$\frac{\delta D}{g(D)} = G_\sigma(\Delta\sigma, R_\sigma) \delta N$$ \[4\]

which gives by integration,

$$\int_0^{D_c} \frac{dD}{g(D)} = G_\sigma(\Delta\sigma, R_\sigma) N_R$$ \[5\]

so that the Wöhler curve is gained as:

$$N_R = \frac{1}{G_\sigma(\Delta\sigma, R_\sigma)} \int_0^{D_c} \frac{1}{g(D)} dD$$ \[6\]

For a block $i$ (made of $n_i$ cycles) of a multi-level loading, each of amplitude $\Delta\sigma_i$ and of stress ratio $R_{\alpha}^i$, one has similarly

$$\int_{D_{i-1}}^{D_i} \frac{dD}{g(D)} = G_\sigma(\Delta\sigma^i, R_{\alpha}^i) n_i = \frac{n_i}{N_{R_i}} \int_0^{D_c} \frac{dD}{g(D)}$$ \[7\]
where the damage at the end (resp. at the beginning) of the block \( i \) is \( D_i \) (resp. \( D_{i-1} \)) and \( N_{Ri} \) is gained from Equation [6]. The sum \( \sum_i \int_{D_i}^{D_{i-1}} \frac{dD}{g(D)} \) equals \( \int_0^{D_c} \frac{dD}{g(D)} \) and Equation [7] unfortunately leads to linear Miner rule \( \sum \frac{n_i}{N_{Ri}} = 1 \).

To obtain a non linear damage accumulation, a more complex modeling is necessary, showing a first limitation (in 1D!) of stress amplitude laws. Other limitations concern the extension of Equation [3] to strain controlled tests and to non cyclic loadings encountered in random fatigue or in seismic applications. Note that the link between a stress and a strain formulation is not clear for nonlinear materials at least as long as no rate written damage model is defined (Lemaitre et al., 1985, Paas, 1990, Lemaitre, 1992, Paas et al., 1993, Peerlings, 1997, Bodin et al., 2002, Lemaitre et al., 2005).

### 3. Damage evolution laws for fatigue

From a thermodynamics point of view, laws function of the current number of cycles \( N \) are not constitutive equations. They incorporate information on the loading and must be theoretically recovered from real constitutive equations, written in a rate form. This makes the modeling not as flexible as for the amplitude laws (the direct introduction the stress ratio is lost) but this way to proceed allows for a natural equivalence between stress and strain formulations and for a natural extension to 3D.

#### 3.1. Lemaitre’s damage law

Lemaitre’s damage law models damage growth governed by plasticity (through the accumulated plastic strain \( p \)) and enhanced by the stress level and triaxiality (through the elastic energy density denoted here \( Y \)). It is written in a rate form as:

\[
\dot{D} = \left( \frac{Y}{S} \right)^s \dot{p}
\]

and introduces 2 damage parameters: the damage strength \( S \) and the damage exponent \( s \). Noting that in tension \( Y = \sigma^2 / 2E \) and making the (small) assumption \( |\sigma| \equiv \sigma_{Max} \), the maximum stress, when damage occurs, allows to derive the damage increment per cycle as:

\[
\frac{\delta D}{\delta N} = \int_{\text{1 cycle}} \dot{D} \, dt \approx \left( \frac{\sigma_{Max}^2}{2ES} \right)^s \int_{\text{1 cycle}} \dot{p} \, dt
\]

or

\[
\frac{\delta D}{\delta N} = \left( \frac{\sigma_{Max}^2}{2ES} \right)^s 2 \Delta \epsilon_p
\]
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with $\Delta \epsilon_p$ the plastic strain amplitude. A cyclic plasticity law $\Delta \epsilon_p(\Delta \sigma)$, eventually derived from the time integration of a kinematic hardening law, gives finally $\frac{\delta D}{\delta N}$ as a function of the stress amplitude. For the power law $\Delta \sigma = K_{\text{cyc}}(\Delta \epsilon_p)^{1/q}$, it is:

$$\frac{\delta D}{\delta N} = \frac{2(\Delta \sigma)^{2s+q}}{(8ES)^sK_{\text{cyc}}^2}$$

[11]

with then a number of cycles to rupture

$$N_R = \frac{(8ES)^sK_{\text{cyc}}^2D_c}{2(\Delta \sigma)^{2s+q}}$$

[12]

where $-(2s + q)$ is the (negative) slope of the Wöhler curve in a log-log diagram.

The stress triaxiality effect of a damage growth enhanced by large hydrostatic stresses $\sigma_{H \text{Max}}$ (compared to von Mises $\sigma_{eq \text{Max}}$) is also naturally derived as in 3D,

$$Y_{\text{Max}} = \frac{\sigma_{eq \text{Max}}^2 R_{\nu}}{2E} \quad R_{\nu} = \frac{2}{3}(1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2$$

[13]

introducing the triaxiality function $R_{\nu}$ (Lemaitre et al., 1985),

$$\frac{\delta D}{\delta N} = \left( \frac{\sigma_{eq \text{Max}}^2 R_{\nu}}{2ES} \right)^s 2\Delta \rho$$

[14]

with $\Delta \rho = \Delta p(\Delta \sigma_{eq})$ the accumulated plastic strain increment over half a cycle. In shear $R_{\nu} = R_{\nu}^{\text{shear}} = \frac{4}{3}(1 + \nu)$, in tension-compression $R_{\nu} = R_{\nu}^{\text{TC}} = 1$, in equi-biaxial tension $R_{\nu} = R_{\nu}^{\text{biax}} = 2(1 - \nu)$, leading to $R_{\nu}^{\text{shear}} < R_{\nu}^{\text{TC}} < R_{\nu}^{\text{biax}}$ and to increasing numbers of cycles to rupture as the stress triaxiality decreases,

$$N_{R}^{\text{biax}} < N_{R}^{\text{TC}} < N_{R}^{\text{shear}}$$

at identical von Mises stress

[15]

with, still for a given applied maximum von Mises stress, the number of cycles to rupture in multiaxial related to $N_{R}^{\text{TC}}$ in uniaxial as:

$$N_{R} = N_{R}^{\text{TC}} R_{\nu}^{-s}$$

[16]

Last but not the least, damage in fatigue of metals usually does not initiates instantaneously. In monotonic tension, the material can undergo plasticity up to a plastic strain threshold $\epsilon_p = \epsilon_{pD}$ of the order of 10 to 30% (i.e. often up to hardening saturation). In fatigue, it can take a few hundreds of percent of accumulated plastic strain before damage initiation (at the damage threshold $p = p_D$). Such a loading dependency of the damage threshold $p_D$ can be represented by considering that damage initiates when the energy density stored by hardening $w_s$ reaches the energetic damage threshold $w_D$, loading independent, which is the amount of energy needed for the incubation of defects (Lemaitre et al., 2000, Lemaitre et al., 2005). According to plasticity framework, the stored energy density is defined as the integral

$$w_s = \int_0^t (\sigma_{eq} - \sigma_y) \dot{\rho} \, dt$$

[17]
with \( \sigma_y \) the yield stress. For monotonic hardening, considering the stress saturated at the value \( \sigma_u \), the ultimate stress, the relationship between plastic strain damage threshold \( \epsilon_{pD} \) and stored energy damage threshold simply reads

\[
\omega_D = \int_0^{\epsilon_{pD}} (\sigma_{eq} - \sigma_y) \, dp \approx (\sigma_u - \sigma_y) \epsilon_{pD} \tag{18}
\]

For fatigue, making the simplifying assumption \( \sigma_{eq} \approx \sigma_{eq \, max} \) during the plastic part of the stress-strain cycle gives the damage threshold \( p_D \), loading dependent, as the solution of:

\[
\omega_D = \int_0^{p_D} (\sigma_{eq \, max} - \sigma_y) \, dp = (\sigma_{eq \, max} - \sigma_y) p_D \tag{19}
\]

or:

\[
p_D = \epsilon_{pD} \left( \frac{\sigma_u - \sigma_y}{\sigma_{eq \, max} - \sigma_y} \right) \tag{20}
\]

recently generalized into \( p_D = \epsilon_{pD} \left( \frac{\sigma_u - \sigma_y}{\sigma_{eq \, max} - \sigma_y} \right)^m \) with \( m \) a material parameter (Lemaitre et al., 2005).

The number of cycles to rupture is then the sum of two terms, the number of cycles to damage initiation \( N_D = N(p = p_D) \) and the number of cycles during which takes place the damage process (in fact the term \( N_R \) of previous analysis with no damage threshold). Equation [12] becomes then:

\[
N_R = N_D + \left( \frac{8E \sigma^4}{K_{cy}^3 D_c} \right)^{d+q} \tag{21}
\]

For two-level fatigue loading the consideration of a damage threshold leads to a bi-linear damage accumulation diagram (Figure 1) more in accordance with experiments (Desmorat, 2000, Lemaitre et al., 2005).

Lemaitre’s damage law also applies to monotonic, creep and creep-fatigue failures (Lemaitre, 1992, Sermage et al., 2000, Lemaitre et al., 2005). For instance, the plastic strain to rupture in tension is:

\[
\epsilon_{pR} \approx \epsilon_{pD} + \left( \frac{2E \sigma_s}{\sigma_u^2} \right)^a D_c \tag{22}
\]

### 3.2. Quasi-brittle materials

For quasi-brittle materials plasticity is often meaningless so that to consider damage growth governed by the accumulated plastic strain has no sense. Two approaches
are presented next to overcome this difficulty and still in order to consider rate form constitutive equations: Paas approach of a damage governed by an equivalent strain rate, generalized Lemaitre’s approach of a damage governed by the main dissipative mechanism of internal friction encountered in these materials.

### 3.2.1. Paas approach for fatigue

Paas (Paas, 1990) proposed the following damage evolution law:

\[
\dot{D} = C D^\alpha \dot{\varepsilon} \langle \dot{\varepsilon} \rangle
\]  

of a damage governed by an equivalent strain rate \( \dot{\varepsilon} \) (Mazars, 1984, de Vree et al., 1995) and enhanced by:

- the damage level through the term \( D^\alpha \) with \( \alpha \) a material parameter \( 0 \leq \alpha < 1 \),
- the strain (and stress) level through the term \( \dot{\varepsilon}^\beta \) with \( \beta \) a material parameter, directly related to the slope of the fatigue curve.

\( C \) is also a material (damage) parameter. A slightly different expression has been proposed by Peerlings (Peerlings, 1997) to avoid the initial zero damage rate,

\[
\dot{D} = C e^{\alpha D} \dot{\varepsilon}^\beta \langle \dot{\varepsilon} \rangle
\]  

1. \( \langle \cdot \rangle \) denotes the positive part of a scalar.
The growth of damage and the fatigue life can be gained in closed form for the situation of uniaxial loading between a zero minimum strain and a maximum strain equal to the (constant) strain amplitude $\Delta \epsilon$, with the assumption that there is no fatigue limit. A first integration over one cycle (assuming $D$ constant on a cycle) gives the amplitude laws,

\[
\frac{\delta D}{\delta N} = C D^\alpha (\Delta \epsilon)^{\beta+1}/(\beta+1) \quad \text{for Paas law}
\]

\[
\frac{\delta D}{\delta N} = C e^{\alpha D} (\Delta \epsilon)^{\beta+1}/(\beta+1) \quad \text{for Peerlings law}
\]

relations which can then be integrated as done in section 2 yielding the number of cycles to rupture $N_R$ as a function of the strain amplitude,

\[
N_R = \left( \frac{(\beta + 1)D_1^{1-\alpha}}{C(1-\alpha)(\Delta \epsilon)^{\beta+1}} \right) \quad \text{for Paas law}
\]

\[
N_R = \left( \frac{(\beta + 1)(1-\epsilon^{-\alpha D_c})}{\alpha C(\Delta \epsilon)^{\beta+1}} \right) \quad \text{for Peerlings law}
\]

which is in both cases a straight line of slope $-(\beta + 1)$ in the $\log \Delta \epsilon$ vs $\log N_R$ fatigue diagram. Note that due to the multiplicative splitting between the damage enhancing term $CD^\alpha$ or $Ce^{\alpha D}$ and the strain enhancing term $\hat{\epsilon}^\beta$, a linear accumulation is obtained for multi-level loading (Miner rule for two-level fatigue loading).

It is also tempting to apply Paas and Peerlings laws to monotonic loading, leading then to:

\[
\int_0^{D_c} D^{-\alpha} dD = C \int_0^{\epsilon_R} \epsilon^\beta d\epsilon \quad \text{for Paas law}
\]

\[
\int_0^{D_c} e^{-\alpha D} dD = C \int_0^{\epsilon_R} \epsilon^\beta d\epsilon \quad \text{for Peerlings law}
\]

and to the rupture strain corresponding to $D = D_c$,

\[
\epsilon_R = \left( \frac{(\beta + 1)D_1^{1-\alpha}}{(1-\alpha)C} \right)^{1/\beta} \quad \text{for Paas law}
\]

\[
\epsilon_R = \left( \frac{(\beta + 1)(1-\epsilon^{-\alpha D_c})}{\alpha C} \right)^{1/\beta} \quad \text{for Peerlings law}
\]

which is in general not found equal to the measured rupture strain if the fatigue values of the material parameters $C$, $\alpha$, $\beta$ are considered. The Paas approach is then suitable for fatigue, and will face difficulties to model failure due to severe overloads.
3.2.2. Generalized damage law

The damage law [8] can be generalized as (stored energy damage threshold included):

\[
\dot{D} = \left( \frac{Y}{S} \right)^* \dot{\pi} \quad \text{as soon as } w_s = w_D
\]

in order to describe damage growth governed by the main dissipative mechanism (Cantournet et al., 2003, Lemaître et al., 2005). It was plasticity (setting \( \pi = p \)) for metals, elastic strains in previous section, it is internal sliding and friction for most materials (as concrete but as also filled elastomers!). It needs an adequate definition of the cumulative measure of the internal sliding \( \pi \), definition which cannot be given here as it is related to the chosen thermodynamics modeling for the material irreversibilities, for example a model with internal sliding and friction for concrete (section 4.4), Drucker-Prager plasticity model for rocks (see next section and also section 4.4).

Just to make a complete transition with Paas approach for fatigue, remark that setting \( \dot{\pi} = \langle \dot{\gamma} \rangle \) altogether with \( Y = \frac{1}{2} E \dot{\gamma}^2 \) and \( w_D = 0 \) makes Equation [29] become

\[
\dot{D} = \left( \frac{E \dot{\gamma}^2}{2S} \right)^* \langle \dot{\gamma} \rangle = \left( \frac{E}{2S} \right)^s \langle \dot{\gamma} \rangle
\]

which is the case \( C = \left( \frac{E}{2S} \right)^s, \alpha = 0, \beta = 2s \) of Paas and Peerlings laws.

3.3. Application to rocks

Mechanical behavior of rocks is often modeled by use of non associated plasticity such as by the classical Mohr-Coulomb and Drucker-Prager plasticity models leading to a quite simple representation of dilatancy. Again, the definition of an equivalent plastic strain is dependent on the plasticity model, but two somehow natural choices for \( \pi \) must be considered:

(a) the choice of damage governed by the equivalent shear plastic strain built from the deviatoric (shear) plastic strain rate \( \dot{\epsilon}^{pD} = \dot{\epsilon}^p - \frac{1}{3} tr \dot{\epsilon}^p 1 \),

\[
\gamma^p = \int \sqrt{\frac{2}{3} p^D : \epsilon^{pD}} dt
\]

which is equal to the accumulated plastic strain \( p \) of von Mises plasticity.

(b) the choice of damage governed by the hydrostatic irreversible strain,

\[
\epsilon^{v} = tr \epsilon^p
\]

each choice leading to a different generalization of Lemaître’s law:

(a) \( \dot{D} = \left( \frac{Y}{S} \right)^* \dot{\gamma}^p \)

(b) \( \dot{D} = \left( \frac{Y}{S} \right)^* \dot{\epsilon}^v \)
It seems difficult to make one expression prevail instead of another. One could argue that damage is mainly governed by dilatancy and enhanced by the micro-defects opening (in accordance with damage law (b)) but by chance both laws will be found strictly equivalent for Drucker-Prager plasticity (with $S_\gamma$ different from $S_v$, see section 4.4).

### 3.4. Micro-defects closure effect – Mean stress effect

For most materials, compressive loadings are less critical than tensile ones, at least as long as structural instabilities do not take place. This is due to the phenomenon of micro-defects or micro-cracks closure when the material is compressed. The damage rate in compression is then lower than the damage rate in tension. In fatigue, this leads to the so called mean stress effect of the fatigue life reduction for a given stress amplitude but for increasing mean stresses $\sigma = \frac{(\sigma_{\text{Max}} + \sigma_{\text{Min}})}{2}$. This is the stress ratio effect of section 2 as $\sigma = \frac{1}{2}\sigma_{\text{Max}}(1 + R)$ and as then an amplitude law function of $\Delta\sigma$ and $R_{\sigma}$ is equivalently function of $\sigma_{\text{Max}}$ and $\sigma$.

A damage law ensuring a damage rate smaller in compression than in tension (for the same stress or strain level) will naturally model the mean stress effect. This will then be the case with an adequate choice of equivalent strain $\dot{\varepsilon}$ or energy $Y$ and of exponents $\beta$ or $s$. A possibility is to consider the following strain energy release rate density derived from thermodynamics assumptions\(^2\) (Ladevèze \textit{et al.}, 1984, Lemaitre, 1992)

$$Y = \frac{1 + \nu}{2E} \left[ \frac{(\sigma)_{+} : (\sigma)_{+}}{(1 - D)^2} + h \frac{(\sigma)_{-} : (\sigma)_{-}}{(1 - hD)^2} \right]$$

$$- \frac{\nu}{2E} \left[ \frac{(\text{tr} \sigma)^2}{(1 - D)^2} + h \frac{(-\text{tr} \sigma)^2}{(1 - hD)^2} \right]$$

introducing $h$ the microdefects closure parameter ($0 < h < 1$) and rewritten in the simple uniaxial case

$$Y = \frac{(\sigma)^2}{2E(1 - D)^2} + h \frac{(-\sigma)^2}{2E(1 - hD)^2}$$

so that for the same stress or strain level (in absolute value) it is around $h$ times smaller in compression than in tension. Considering Lemaitre’s damage evolution law \([8]\) gives a damage rate even smaller in compression as then:

$$\dot{D}_{\text{compression}} \approx h^s \dot{D}_{\text{tension}} \ll \dot{D}_{\text{tension}}$$

For metals, the damage increment per cycle between $\sigma_{\text{Min}} < 0$ and $\sigma_{\text{Max}} > 0$ is:

$$\frac{\delta D}{\delta N} \approx \frac{1}{(2ES)^s} \left[ \frac{\sigma_{\text{Max}}^2}{(1 - D)^2} + h \frac{\sigma_{\text{Min}}^2}{(1 - hD)^2} \right] \Delta \varepsilon_p$$

\(^2\) $\langle . \rangle_+$ (resp. $\langle . \rangle_-$) standing for the positive (resp. negative) part of a tensor in terms of principal values.
leading to:

\[
\int_0^D \left[ \frac{1}{(1-D)^2} + h \frac{R_s^{2s}}{(1-hD)^2} \right]^{-1} dD = \left( \frac{\Delta \sigma}{(8ES)^s} \right)^2 N \Delta \epsilon_p
\]  \hspace{1cm} [38]

and to the representation of the mean stress effect in fatigue as:

\[
N_R = \frac{(8ES)^s K_{cyc}^q}{(\Delta \sigma)^{2s+q}} \int_0^{D_c} \left[ \frac{1}{(1-D)^2} + h \frac{R_s^{2s}}{(1-hD)^2} \right]^{-1} dD
\]  \hspace{1cm} [39]

3.5. Damage post-processing

For early design of mechanical components, the coupling of the strain behavior with the damage may be neglected and a post-processing of damage evolution is possible after a classical structure analysis (Hayhurst et al., 1973). This approach is an uncoupled analysis, it is based on a reference nonlinear computation allowing without damage representation for a correct estimation of the strain energy density and of either the accumulated plastic strain \( p \) or the cumulative measure of internal sliding \( \pi \). This analysis is efficient for ductile materials as metals (Lemaitre, 1992) or elastomers (Cantournet et al., 2003). It needs to be developed for other materials but note that it probably won’t easily apply to concrete for which damage itself is the main cause of the stress-strain (softening) response so that the above reference calculation does not exist! In last case (when the coupling between strains and damage is strong) and also for accurate engineering applications a fully coupled analysis is necessary. The constitutive equations of the full damage model (see section 4) need to be implemented within a Finite Element computer code.

The post-processing consists in calculating the integral

\[
D = \int_{t_D}^t \left( Y(t) \frac{S}{S} \right)^s \tilde{\pi}(t) dt
\]  \hspace{1cm} [40]

from results of classical computations made without damage; it corresponds to the \textit{a posteriori} time integration of the damage evolution law \[8\] or \[29\] or \[33\] where \( t_D \) is the time at damage initiation, for plastic materials solution of \( p_D = \int_0^{t_D} \dot{\pi}(t) dt \), more generally of \( w_D = \int_0^{t_D} \tilde{w}_s(t) dt \) (see section 3.1).

The time histories of the measure of internal sliding \( \pi \) (often \( p \)), which needs a \textit{a priori} definition or a constitutive modeling, and of the energy density release rate \( Y \) are the necessary inputs for the damage post-processing. The time to mesocrack initiation \( t_R \) or, for cyclic loading, the corresponding number of cycles to mesocrack initiation \( N_R \) corresponds to the reach of the critical damage \( D_c \) at the structure most loaded point.

Note that the post-processing may take place at one point only, if known: the most loaded point, usually where \( Y \) is maximum. Rewritten \( Y = \frac{\sigma^*}{2E} \) it is also
where the damage equivalent stress \( \sigma^* = \sigma_{eq} \sqrt{R_1/2} \) is maximum. The post-processing may also take place at different (chosen) Gauss points of a structure. In any case, the damage law being written in a rate form, the post-processing applies to complex loading, not necessary 1D nor cyclic, nor even isothermal.

3.6. Jump-in-cycles procedure

For fatigue loading periodic or periodic by blocks the previous calculations, step by step in time, becomes prohibitive when the number of cycles becomes large (\(10^4\), \(10^5\), \(10^6\), . . . ). For that reason a simplified numerical method is needed which allows to “jump” full blocks of \(\Delta N\) cycles. The time integration of Equation [40] or the 3D Finite Element computation of a structure is performed over one cycle once in a while and the computation time may be almost divided by \(\Delta N\).

This “jump” in cycles procedure works as follows (Lemaitre et al., 1994).

1) Before any damage growth, i.e. up to the threshold \(w_s = w_D\), run the computation until a stabilized cycle \(N_s\) is reached and let \(\frac{\delta \pi}{\delta N} \bigg|_{N_s}\) (resp. \(\frac{\delta p}{\delta N} \bigg|_{N_s}\)) be the accumulated internal sliding (resp. plastic strain) increment over this single cycle. Assume then that during the number \(\Delta N\) of the next cycles, \(\pi\) or \(p\) remains linear versus the number of cycles \(N\) and calculate the number of cycles to be jumped \(\Delta N\) cycles as

\[
\Delta N = \frac{\Delta \pi}{\frac{\delta \pi}{\delta N} \bigg|_{N_s}} \quad \text{or} \quad \Delta N = \frac{\Delta p}{\frac{\delta p}{\delta N} \bigg|_{N_s}}
\]

[41]

where \(\Delta \pi\) is a given value which determines the accuracy of the procedure. For plastic materials, \(\Delta \pi \approx p_D/50\) is a good compromise between accuracy and time cost. The accumulated plastic strain is updated as:

\[
\pi(N_s + \Delta N) = \pi(N_s) + \Delta \pi \quad \text{or} \quad p(N_s + \Delta N) = p(N_s) + \Delta p
\]

[42]

The stresses, strains, irreversible strains at the end of the cycle \(N_s\) and the cumulative measure of internal sliding \(\pi(N_s + \Delta N)\) (resp. the accumulated plastic strain \(p(N_s + \Delta N)\)) are then the initial values for the computation of the first time increment of the cycle \((N_s + \Delta N + 1)\).

Repeat the jumps up to the occurrence of damage.

2) Once damage occurs, run first the computation at constant damage until a new stabilized cycle \(N_s\) is reached. Calculate

- the increment \(\frac{\delta \pi}{\delta N} \bigg|_{N_s}\) or \(\frac{\delta p}{\delta N} \bigg|_{N_s}\)

- the damage increment \(\frac{\delta D}{\delta N} \bigg|_{N_s}\)
Assume again that during the number $\Delta N$ of the next cycles $\pi$ or $p$ and $D$ remain linear versus $N$ and calculate the number of cycles to be jumped as

$$\Delta N = \min \left( \frac{\Delta p}{\frac{\partial \pi}{\partial N}|_{N_s}}, \frac{\Delta D}{\frac{\partial D}{\partial N}|_{N_s}} \right)$$

where $\Delta D$ is a given value which determines the accuracy on the damage. Here again $\Delta D \approx D_c/50$ is a good compromise between accuracy and time cost and take $\Delta p = (S_{Y\text{Max}}) \Delta D$ with $Y_{\text{Max}}$ the maximum value of the energy density $Y$ over the cycle $N_s$.

The accumulated internal sliding or plastic strain and the damage are finally updated as

$$\pi(N_s + \Delta N) = \pi(N_s) + \frac{\partial \pi}{\partial N}|_{N_s} \Delta N$$

or

$$p(N_s + \Delta N) = p(N_s) + \frac{\partial p}{\partial N}|_{N_s} \Delta N$$

and

$$D(N_s + \Delta N) = D(N_s) + \frac{\partial D}{\partial N}|_{N_s} \Delta N$$

The stresses, strains, plastic strains, at the end of the cycle $N_s$, the accumulated internal sliding $\pi(N_s + \Delta N)$ or plastic strain $p(N_s + \Delta N)$ and the damage $D(N_s + \Delta N)$ are then the initial values for the computation of the first time increment of the cycle $(N_s + \Delta N + 1)$.

4. Toward an unified modeling for damage and fatigue?

The idea of an unified damage model for many materials is made possible by the consideration of the generalized damage law [29], i.e. by assuming that damage is governed by the main dissipative mechanism, often internal sliding and friction, through the cumulative measure $\pi$. This kind of modeling applies to metals (Lemaitre et al., 1985, François et al., 1993) for which internal slips are mainly due to dislocations creation and evolution, but also to non metallic materials such as concrete and elastomers (Desmorat et al., 2001, Cantournet et al., 2003, Lemaitre et al., 2005, Desmorat et al., 2006).

The present section is a synthetic thermodynamics presentation of in fact different constitutive models. For more details on the general damage mechanics thermodynamics framework refer to (Lemaitre et al., 1985) or (Lemaitre et al., 2005).
4.1. Thermodynamics Variables

Define $\varepsilon^p$, $a$, $q$ and $D$ as internal variables associated with $-\sigma^p$, $x$, $Q$ and $-Y$ (see table 1). The physical meaning of these thermodynamics variables depends on the type of material and of the physical dissipative mechanisms (see section 4.4). The variable $\varepsilon^p$ stands for the irreversible strains. The couples of variables $(Q, q)$ and $(x, a)$ will represent hardening, isotropic and kinematic, of ductile materials, they will represent consolidations, also isotropic and kinematic, for quasi-brittle materials. $D$ is the damage variable and the energy density $Y$ is the strain energy release rate density.

Table 1. State and associated variables

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>Type</th>
<th>State variables</th>
<th>Observable</th>
<th>Internal</th>
<th>Associated variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elasticity</td>
<td>tensor</td>
<td>$\varepsilon$</td>
<td>$\varepsilon^p$</td>
<td>$\sigma^p$</td>
<td></td>
</tr>
<tr>
<td>Internal</td>
<td>tensor</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$x$</td>
<td></td>
</tr>
<tr>
<td>sliding</td>
<td>tensor</td>
<td>$q$</td>
<td>$Q$</td>
<td>$Q$</td>
<td></td>
</tr>
<tr>
<td>Damage</td>
<td>scalar</td>
<td>$D$</td>
<td>$-Y$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.2. State and evolution laws: generalized damage model

Use the general form of the state potential (Helmholtz free energy):

$$\rho \psi = (1 - D) [W_1(\varepsilon) + W_2(\varepsilon - \varepsilon^p)] + w_s(q, a)$$

where $W_1$ and $W_2$ define the strain energy density and $w_s = G(q) + \frac{1}{2}C_x a : a$ the stored energy density ($C_x$ is a material parameter), function of the scalar variable $q$ and of the tensorial variable $a$. The state laws are:

$$\sigma = \rho \frac{\partial \psi}{\partial \varepsilon} = (1 - D) \frac{\partial (W_1 + W_2)}{\partial \varepsilon}$$

$$\sigma^p = -\rho \frac{\partial \psi}{\partial \varepsilon^p} = (1 - D) \frac{\partial W_2}{\partial \varepsilon}$$

$$x = \rho \frac{\partial \psi}{\partial a} = \frac{\partial w_s}{\partial a} = C_x a$$

$$Q = \rho \frac{\partial \psi}{\partial q} = \frac{\partial w_s}{\partial q} = G'(q)$$

$$Y = -\rho \frac{\partial \psi}{\partial D} = W_1 + W_2$$
They naturally define the effective stresses $\tilde{\sigma}, \tilde{\sigma}^\pi$ such as the elasticity law written in terms of strains and of effective stresses does not depend explicitly upon $D$ (strain equivalence principle):

$$\tilde{\sigma} = \frac{\sigma}{1 - D} = \frac{\partial (W_1 + W_2)}{\partial \varepsilon}$$

$$\tilde{\sigma}^\pi = \frac{\sigma^\pi}{1 - D} = \frac{\partial W_2}{\partial \varepsilon}$$

[47]

Use as dissipative potential:

$$F = f + F_x + F_D$$

[48]

where:

$ - f = ||\tilde{\sigma}^\pi - x|| - Q - \sigma_s < 0$ defines the reversibility domain, $||.||$ is a norm in the stresses space (not necessary von Mises norm) and $\sigma_s$ is the reversibility limit,

$ - $ the functions $F_x = \frac{\gamma}{2C_x} : x$ and $Q = Q(q) = G'(q)$ model the internal sliding nonlinearity. $C_x$ and $\gamma$ are the kinematic hardening or consolidation parameters,

$ - F_D = \frac{S}{(s+1)(1-D)} \left( \frac{Y}{S} \right)^{s+1}$ is Lemaitre’s damage potential with $S$ and $s$ the damage parameters. Note that another classical choice, for instance for quasi-brittle elasticity, is Marigo damage potential $f = F_D = Y - \kappa(D)$ (Marigo, 1981). But such a choice does not apply to fatigue.

The evolution laws derive from the dissipative potential through the normality rule,

$$\dot{\varepsilon}^\pi = \mu \frac{\partial F}{\partial \tilde{\sigma}^\pi} = \frac{\mu}{1 - D} \frac{\tilde{\sigma}^\pi - x}{||\tilde{\sigma}^\pi - x||}$$

$$\dot{q} = -\mu \frac{\partial F}{\partial Q} = \dot{\mu}$$

$$\dot{a} = -\mu \frac{\partial F}{\partial \varepsilon} = \mu \frac{\partial F}{\partial x} = (1 - D) \dot{\varepsilon}^\pi - \gamma a \dot{q}$$

$$\dot{D} = \mu \frac{\partial F}{\partial Y} = \mu \left( \frac{Y}{S} \right)^s = \left( \frac{Y}{S} \right)^s \dot{\tau}$$

[49]

with $\dot{\mu}$ a Lagrange multiplier given by the consistency condition $f = 0$ and $\dot{f} = 0$ for non viscous materials or given by a viscosity law such as generalized Norton’s law $\dot{\tau} = (f/K_N)^N$ for viscous materials. The multiplier $\dot{\mu}$ is equal to a norm of the inelastic strain rate as first equation of [49] leads to:

$$\frac{\dot{\mu}}{1 - D} = \frac{\dot{q}}{1 - D} = ||\dot{\varepsilon}^\pi||$$

[50]

This defines the expected cumulative measure $\pi$ of the internal sliding,

$$\pi = \int_0^t ||\dot{\varepsilon}^\pi|| \, dt$$

[51]
which gives back the accumulated plastic strain as
\[
p = \int_0^t \|\dot{\varepsilon}^p\|_{\text{von Mises}} \, dt = \int_0^t \sqrt{\frac{2}{3} \dot{\varepsilon}^p : \dot{\varepsilon}^p} \, dt \tag{52}
\]

in von Mises plasticity.

The generalized damage evolution law derives from the last equation of \([49]\):
\[
\dot{D} = \left(\frac{Y}{S}\right) \dot{\pi} \quad \text{if} \quad \pi > \pi_D \tag{53}
\]

where \(\pi_D\) is the damage threshold, generally equal to zero for quasi-brittle materials. Written \(\dot{D} = (Y/S)\dot{\pi}\), it models a damage governed by the main dissipative mechanism, here internal sliding and friction (through \(\dot{\pi}\)), and enhanced by the value of the strain energy density (through \(Y\)). Written \(\dot{\pi} = (S/Y)\dot{D}\), it models the increasing internal sliding due to damage accumulation. As the constitutive equations describing both damage and internal sliding are fully coupled, the interpretation is of course a combination of both points of view: each phenomenon, damage or internal sliding, enhances the other one.

4.3. Positivity of the intrinsic dissipation

The generalized damage model satisfies the positivity of the intrinsic dissipation \(D = \sigma : \dot{\varepsilon} - \rho \psi\) as
\[
D = \sigma^p : \dot{\varepsilon}^p - Q \dot{\gamma} - \mathbf{x} : \dot{\mathbf{a}} + Y \dot{D} \\
= \left(\sigma^p : \frac{\partial F}{\partial \sigma^p} + Q \frac{\partial F}{\partial Q} + \mathbf{x} : \frac{\partial F}{\partial \mathbf{x}} + Y \frac{\partial F}{\partial Y}\right) \dot{\mu} \geq 0 \tag{54}
\]

when the dissipation potential \(F(\sigma^p, Q, \mathbf{x}, Y; D)\) is a non negative convex function of its arguments \(\sigma^p, Q, \mathbf{x}, Y\) with \(F(\mathbf{0}, 0, \mathbf{0}, 0; D) = 0\) and where the damage \(D\) acts as a parameter. Using the evolution laws, the intrinsic dissipation may also be rewritten:
\[
D = \left(\sigma_s + \sigma_v + \frac{\gamma}{C_s} \mathbf{x} : \mathbf{x}\right) (1 - D) \|\dot{\varepsilon}^p\| + Y \dot{D} \geq 0 \tag{55}
\]

with \(\sigma_v\) the viscous stress (vanishing for non viscous materials) such as \(f = \sigma_v\) for viscous internal sliding.
Figure 2. Cyclic tensile curve of a filled SBR (model: solid line, experiment: dash line)

Figure 3. Fatigue curve of a filled SBR for a mean elongation $\lambda_{\text{moy}} = 2.53$

4.4. Application to metals, elastomers, quasi-brittle materials and rocks

For metals, $W_1 = 0$, $W_2$ is quadratic, $Q(q) = R(r)$ is the isotropic hardening, $x = X$ the kinematic hardening, $||.||$ is the von Mises norm $(.)_{eq}$, $\varepsilon^p$ is the plastic strain $\varepsilon^p$ and the generalized damage law [53] recovers Le-maitre damage evolution law [8] where $\pi$ equals the accumulated plastic strain $p$. 
For filled elastomers, $W_1$ is an hyperelasticity density (Mooney, 1940, Hart-Smith, 1966, Lambert-Diani et al., 1999), $Q = 0$, $W_2$ is chosen quadratic (Desmorat et al., 2001). The variable $\mathbf{x}$ stands for the residual micro-stresses due to internal sliding with friction of the macro-molecular chains on themselves and on the black carbon filler particles. The internal inelastic strain $\mathbf{\epsilon}^\pi$ represents the homogenized displacements incompatibilities due to friction at microscale. Interesting feature, the parameter $\gamma$ of the nonlinear evolution law for $\mathbf{x}$ is responsible for the cyclic stress softening encountered in these materials (Müllins effect, Figure 2). An example of calculated fatigue curve of the applied elongation amplitude $\Delta \lambda$ (finite strain framework) versus the number of cycles to rupture $N_R$ is given in figure 3. The measured elongation to rupture in tension $\lambda_R = 7.2$ is also reported on the diagram (it corresponds to 620% of deformation), so that such a modeling applies for monotonic and fatigue failures.

For quasi-brittle materials such as concrete, take $W_1$ and $W_2$ quadratic, neglect for fatigue applications the isotropic consolidation ($Q = 0$). The model represents then the stress-strain response as well as the hysteresis of concrete (Figure 4), not perfectly for the hysteresis loops because of the simple modeling of the consolidations ($Q = 0$, $\mathbf{x}$ linear) but well enough to envisage structural applications (Desmorat et al., 2006, Ragueneau, 2006), even in earthquake engineering.

![Figure 4. Hysteresis response of concrete in compression](image)

The proposed model allows for the step by step computation of the material fatigue curves. An example of calculated curve is given in Figure 5 as the normalized stress $\sigma_{Max}/f_c$ vs number of cycles to rupture $N_R$ (with $f_c$ the compressive strength). The model corresponds to the dot lines, black for the symmetric loading, grey for an $\epsilon_{min} = 0$ loading. The results for $\epsilon_{min} = 0$ give a lower bound of the experimental data for concrete tested in fatigue with a zero stress ratio $R_\sigma$ (Grzybowski et al.,...
Damage and fatigue

1993, Paskova et al., 1997) and seem to be conservative. The computations give the same tendencies than the simple Aas-Jakobsen formula (Aas-Jackobsen et al., 1973) function of the stress ratio $R_\sigma = \sigma_{\min}/\sigma_{\max}$ and taking then into account the mean stress effect,

$$\frac{\sigma_{\max}}{f_c} = 1 - \beta(1 - R_\sigma) \log N_R$$

The material parameter $\beta$ set to 0.1 allows to recover the fatigue curves for both stress ratios (stress ratio taken in the formula simply as $R_\sigma = -1$ for the symmetrical loading, to zero for the $\epsilon_{\min} = 0$ case). As the damage model does not represent the material behavior dissymmetry, the analysis is mainly qualitative. One can nevertheless notice that a mean stress effect is reproduced and that the value obtained from the computations is found of the order of magnitude of the usual value $\beta = 0.0685$ for light concrete (Destrebecq, 2005).

Figure 5. Normalized fatigue curves – Comparison between model (dot lines), experiments (marks) and Aas-Jakobsen formula (straight lines)

For rocks, the modeling depends mainly on the ductility observed. Brittle materials need a $W_1$ term, when to consider a more classical plasticity model for ductile ones is often sufficient. The main difficulty concerns the choice of the norm $||.||$ in the yield function $f$, choice directly related to the modeling of dilatancy, i.e. of positive permanent hydrostatic stresses (dilatancy) in compression (with the convention of positive stresses or strains in tension). As an example, let us derive next the damage model for Drucker-Prager plasticity (case $W_1 = 0$). Taking $W_2$ quadratic, $\epsilon^p = \epsilon^p$ as plastic strain and setting $C_x = \frac{2}{3}C$ as often for deviatoric kinematic hardening gives ($K$ is Hooke’s tensor):

\[ \frac{\sigma_{\max}}{f_c} = 1 - \beta(1 - R_\sigma) \log N_R \]
\[
\sigma = \frac{\sigma}{1 - D} = E : (\varepsilon - \varepsilon^p)
\]
\[
x = \frac{2}{3} C a
\]
\[
Q = G'(q)
\]
\[
Y = \frac{1}{2}(\varepsilon - \varepsilon^p) : E : (\varepsilon - \varepsilon^p) = \frac{\sigma_{eq}^2 R_v}{2E(1 - D)^2}
\]

with \( R_v = \frac{2}{3}(1 + \nu) + 3(1 - 2\nu)(\frac{2\mu}{\sigma_{eq}})^2 \) the triaxiality function (Equation 13). Use as criterion and dissipation functions (\( \cdot \)eq is von Mises norm, angles \( \beta \) and \( \psi \) are material parameters):
\[
f = (\tilde{\sigma} - x)_{eq} + \tan \beta \text{ tr } \tilde{\sigma} - Q - \sigma_s
\]
\[
F = f + (\tan \psi - \tan \beta) \text{ tr } \tilde{\sigma} + F_x + F_D
\]
gives the evolution laws of Drucker-Prager plasticity coupled with damage,
\[
\dot{\varepsilon}^p = \dot{\pi} \left[ \frac{3}{2} (\tilde{\sigma} D - x)_{eq} + \tan \psi \right]
\]
\[
\dot{\pi} = \frac{\dot{q}}{1 - D}
\]
\[
\dot{x} = (1 - D) \left[ \frac{2}{3} C \dot{\varepsilon}^{pD} - \gamma x \dot{\pi} \right]
\]
\[
\dot{D} = \left( \frac{Y}{S} \right) \dot{\pi}
\]

and to take the von Mises deviatoric norm of first equation of [59] gives \( \pi \) as simply the shear equivalent plastic strain,
\[
\pi = \gamma_p = \int \sqrt{\frac{2}{3} \varepsilon^{pD} : \varepsilon^{pD}} dt
\]

To take the trace of first equation of [59] gives \( \dot{\varepsilon}^p = \text{ tr } \dot{\varepsilon}^p = 3 \tan \psi \dot{\pi} \) and:
\[
\pi = \frac{\varepsilon^p}{3 \tan \psi}
\]

so that both generalizations (a) and (b) of Lemaitre’s law (Equation 33) are valid altogether with \( S = S_\gamma = S_v/(3 \tan \psi)^{1/\nu} \).

5. High cycle fatigue from damage analysis

Last, let us detail the possibility to use damage models to represent high cycle fatigue failure conditions, i.e. mainly fatigue in an elastic regime. In metals, plasticity
then does not occur at the mesoscale of the Representative Volume Element (RVE) of continuum mechanics but at a microscale, the defects scale. A two-scales damage model has been built based on this feature (Lemaitre et al., 1994). It is a locally coupled analysis, eventually with initial plastic strain and damage, which can be performed by post-processing an elastic computation. The jump-in cycles procedure of section 3.6 can advantageously be used.

5.1. Two scales damage model for metals

The two scales under consideration are represented in Figure 6. The mesoscopic scale or mesoscale is the scale of the RVE of continuum mechanics, the microscopic scale or microscale is the scale of the microdefects. At the mesoscale, the stresses are denoted \( \sigma \), the total, elastic and plastic strains \( \epsilon, \epsilon^e, \epsilon^p \). They are known from a thermoelastic Finite Element computation with usually for high cycle fatigue \( \epsilon^p \approx 0 \). The values at the microscale have an upper-script \( \mu \). Plasticity and damage are assumed to occur at the microscale only, \( \epsilon^p \mu \neq 0, D \geq 0 \), where for simplicity the damage variable at the microscale has no upper-script \( (D = D^\mu) \).

In order to illustrate the naturally ability of continuous damage to deal with complex loading, anisothermal conditions are assumed next so that the material parameters are temperature dependent and the thermal expansion, omitted so far, has to be introduced. The thermoelastic law for the RVE reads:

\[
\epsilon = \frac{1 + \nu}{E} \sigma - \nu \frac{1}{E} \text{tr} \sigma 1 + \alpha(T - T_{\text{ref}})1 \tag{62}
\]

with \( E \) the Young modulus, \( \nu \) the Poisson ratio, \( \alpha \) the thermal expansion coefficient and \( T_{\text{ref}} \) the reference temperature. The temperature field in the structure is usually determined from an initial heat transfer computation. A law of thermo-elasto-plasticity coupled with damage is considered at microscale. The elasticity law reads then (recall that \( \mu \)-upper-script stands for “variable at microscale”):

\[
\epsilon^{\mu e} = \frac{1 + \nu}{E} \frac{\sigma^{\mu}}{1 - D} - \nu \frac{1}{E} \frac{\text{tr} \sigma^{\mu}}{1 - D} 1 + \alpha^{\mu}(T - T_{\text{ref}})1 \tag{63}
\]
where the thermal expansion coefficient $\alpha^\mu$ is taken next equal to the meso coefficient $\alpha$. In the yield criterion, the hardening $X^\mu$ is kinematic, linear, and the yield stress is the asymptotic fatigue limit of the material, denoted $\sigma^\infty_f$,

$$f^\mu = (\bar{\sigma}^\mu - X^\mu)_{eq} - \sigma^\infty_f$$  \[64\]

where $\bar{\sigma}^\mu = \sigma^\mu / (1 - D)$ is the effective stress and where $f^\mu < 0 \implies$ elasticity. A linear kinematic hardening $X^\mu$ is considered at microscale ($F^\mu_{x} = 0$). The set of constitutive equations at microscale is then:

$$\begin{align*}
\epsilon^\mu &= \epsilon^{\mu e} + \epsilon^{pp} \\
\epsilon^{\mu e} &= \frac{1 + \nu}{E} \sigma^\mu - \frac{\nu}{E} tr \bar{\sigma}^\mu 1 + \alpha(T - T_{ref}) 1 \\
\bar{\sigma}^\mu &= \frac{E}{1 + \nu} \\
\dot{\epsilon}^{pp} &= \frac{3}{2} (\bar{\sigma}^\mu - X^\mu)_{eq} \dot{\bar{\sigma}}^\mu \\
\frac{d}{dt} \left( \frac{X^\mu}{C_y} \right) &= \frac{2}{3} \dot{\epsilon}^{pp} (1 - D) \\
\dot{D} &= \left( \frac{Y^\mu}{S} \right) \dot{p}^\mu \text{ if } p^\mu > p_D \\
D &= D_c \implies \text{crack initiation}
\end{align*}$$ \[65\]

with the plastic modulus $C_y$, the damage strength $S$, the damage exponent $s$ and the critical damage $D_c$ as material parameters. The damage evolution (last equation of previous set) is lower in tension than in compression due to the consideration of the micro-defects closure parameter $h$ within $Y^\mu$ as usually for metals $h \approx 0.2$ (Lemaitre, 1992) and:

$$Y^\mu = \frac{1 + \nu}{2E} \left[ \frac{\langle \sigma^\mu \rangle_+ \cdot \langle \sigma^\mu \rangle_+}{(1 - D)^2} + h \frac{\langle \sigma^\mu \rangle_- \cdot \langle \sigma^\mu \rangle_-}{(1 - h D)^2} \right]$$

$$- \frac{\nu}{2E} \left[ \frac{(tr \sigma^\mu)^2}{(1 - D)^2} + h \frac{(tr \sigma^\mu)^2}{(1 - h D)^2} \right]$$  \[66\]

$p^\mu$ is the accumulated plastic strain at microscale and $p_D$ the damage threshold (for loading dependent threshold see section 3.1 and refer to (Lemaitre et al., 2005)). The plastic multiplier $\dot{\lambda} = \dot{p}^\mu (1 - D)$ is determined from the consistency condition $f^\mu = 0$, $\dot{f}^\mu = 0$.

The scale transition meso $\to$ micro can be governed by different laws, the simplest being Lin-Taylor law of a strain at microscale identical to the strain at mesoscale (hypothesis used in initial two scale damage model and DAMAGE-90 post-processor):

$$\epsilon^\mu = \epsilon$$  \[67\]

For isothermal loading, Eshelby-Kröner localization law, based on Eshelby analysis of the spherical inclusion, is preferred (Eshelby, 1957, Kröner, 1961),

$$\epsilon^\mu = \epsilon + b(\epsilon^{pp} - \dot{\epsilon}^p)$$  \[68\]
Eshelby’s analysis which leads, when damage and temperature coupling are taken into account, to the modified (anisothermal) Eshelby-Kröner localization law (Seyedi et al., 2004):

\[
\varepsilon^{\mu D} = \frac{1}{1 - bD} \left[ \varepsilon^{D} + b \left( (1 - D)\varepsilon^{\mu p} - \varepsilon^{p} \right) \right]
\]

[69]

\[
\varepsilon^{\mu H} = \frac{1}{1 - aD} \left[ \varepsilon^{H} + a \left( (1 - D)\alpha^{\mu} - \alpha \right) (T - T_{ref}) \right]
\]

[70]

now used in DAMAGE-2005 post-processor, where \( \varepsilon^{H} = \frac{1}{3} tr \varepsilon \) and \( \varepsilon^{\mu H} = \frac{1}{3} tr \varepsilon^{\mu} \) are the hydrostatic strains, respectively at mesososcale and at microscale, and where \( a \) and \( b \) are the Eshelby parameters for a spherical inclusion,

\[
a = \frac{1 + \nu}{3(1 - \nu)}, \quad b = \frac{2}{15} \frac{4 - 5\nu}{1 - \nu}
\]

Figure 7. Computed thermal fatigue curve

The two scale damage model allows for the calculation of Wöhl er curves at different mean stresses (Sermage et al., 1999, Desmorat, 2000, Lemaitre et al., 2005). It also represents thermal fatigue, for example (Figure 7) of a 1D bar blocked at its two extremities and uniformly heated and cooled between a minimum value \( T_{\text{min}} \) and a maximum value \( T_{\text{Max}} \). The numbers of cycles to rupture are gained by use of DAMAGE-2005 post-processor, and still corresponds to \( N_R = N(D = D_c) \). They are plotted in Figure 7 which is then is the computed thermal fatigue curve of the considered steel, \( T_{\text{Max}} \) vs \( N_R \). Written in a rate form, the model allows to deal with complex thermo-mechanical loading, where both the meso-strains or stresses and the temperature vary. As an illustration, Figure 7 represents the thermal fatigue response of a pipe subject to both temperature \( T(t) \) and internal pressure \( P(t) \) variations. The cylinder is assumed thin, the temperature uniform. The stresses due to the thermal loading and to the mechanical loading are of the same order of magnitude.

Different pressure loadings are considered: 0) constant \( P = P_{\text{Max}} \), 1) in phase with the temperature (case 1): \( P = 0 \) for \( T = T_{\text{min}} \), \( P = P_{\text{Max}} \) for \( T = T_{\text{Max}} \), and
2) out of phase with the temperature (case 2): \( P = 0 \) for \( T = T_{\text{Max}} \), \( P = P_{\text{Max}} \) for \( T = T_{\text{min}} \). The results are given in Figure 8 where \( \Delta T = T_{\text{Max}} - T_{\text{min}} \) is the temperature amplitude. A strong effect of temperature-pressure out of phase loading on the number of cycles to rupture is obtained, comforting us in the idea that one must develop adequate tools for anisothermal thermomechanical cases.

5.2. **Toward a two scale damage model for quasi-brittle materials**

The same two scale damage modeling still must be developed for high cycle fatigue of quasi-brittle materials. There are conceptually no difficulty but practically things are different:

- a new localization law must be derived, *i.e.* also justified for the microstructure and the degradation mechanisms encountered in the considered material,
- a 3D (single scale) damage model must have proven efficient for those materials. The dilatancy must be correctly represented, the dissymmetry tension compression must have been correctly reproduced (Desmorat *et al.*, 2006). This will be the damage model at microscale,
- the number of material parameters must be low! the identification fast!
6. Conclusion

Lemaitre’s damage law has proven efficient for fatigue of metals. It can be generalized into $\dot{D} = (Y/S)^{\pi} \dot{\pi}$ of damage governed by the main dissipative mechanism in order to apply to other materials, as concrete, elastomers or rocks. A few damage parameters are introduced, 2 for damage, the damage strength $S$ and the damage exponent $s$, eventually 1 for a damage growth lower in tension than in compression, the micro-defects closure parameter $h$. Stress or strain amplitude laws as well as the Wöhler curve of materials are calculated from the time integration of the damage law. The stress triaxiality effect is taken into account, the mean stress effect recovered, qualitatively for quasi-brittle materials.

A unified damage model, coupling plastic or internal sliding irreversibilities with damage, can then be build to recover the generalized damage law. It allows for the representation of both the hysteretical and the fatigue responses of materials. Also written in a rate form, it applies to the case of complex loading as non cyclic (seismic, random fatigue...), non isothermal, non proportional loadings. High temperature damage behavior can also be addressed, as creep in the pioneering works of Kachanov (1958) and Rabotnov (1969) and creep-fatigue. More detailed application examples can be found in (Lemaitre et al., 2005) or in present volume (Ragueneau, 2006) for structures.

A lot of nice results have been obtained so far, applications are now needed to acquire engineering experience for designing structures with continuum damage mechanics.

7. Bibliography

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