Stress singularities applied to crack initiation by damage of multimaterial joints

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Abstract

We consider a multimaterial joint consisting of dissimilar layers submitted to plane deformations. The elastic stress tensor field at any corner of this kind of joint often exhibits a stress singularity expressed as a function of the \(r-\theta\) polar coordinates by

\[
\sigma = h \, r^{\delta-1} F(\theta),
\]

where \(h\) is the intensity factor and \(\delta\) the singularity exponent. The resolution of the local singularity problem leads to the closed-form expression of \(F\) when \(\delta\) is obtained as a root of a \(2 \times 2\) determinant. The exponent \(\delta\), always larger than 0 and smaller than 1, leads to a finite strain energy which is used as an input in a damage law. The damage rate \(D\), where \(D\) represents the surface density of microcracks or microvoids, is a function of the strain energy density release rate \(Y\). Some examples of crack initiation conditions are pointed out for brittle or quasi-brittle materials for joints loaded in monotonic loading or in fatigue. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Fracture always starts at stress concentration points which are often stress singularities at corners for elastic materials (Williams, 1952; Kondrat’ev, 1967; Stern and Soni, 1976; Grisvard, 1989), or at joints consisting of anisotropic layers (Leguillon and Sanchez-Palencia, 1987; Desmorat, 1996; Desmorat and Leckie, 1998). The main result of these studies is that the singular field for displacements \(u\) and stresses \(\sigma\) in elasticity can be expressed as a function of the polar coordinates \(r-\theta\) (vectors and tensors are represented by bold characters):

\[
\begin{align*}
\mathbf{u} &= h \, r^{\delta} \mathbf{g}(\theta), \\
\mathbf{\sigma} &= h \, r^{\delta-1} \mathbf{F}(\theta),
\end{align*}
\]

where the singularity exponent \(\delta\) is the solution of an eigenvalue problem defined by the boundary conditions close to the tip. The intensity factor \(h\) of the problem depends on the external loads and the geometry of the structure.

Other studies take into account the plastic behavior of the material (Hutchinson, 1968; Rice and Rosengren, 1968; Rudge, 1993; Wei and Wang, 1995), but plasticity with saturating hardening and/or damageable behavior erases the singular field.

The singular elastic solution is not physically possible but contains valuable information about
the energy concentrated close to the tip. In the present work, we will first recall the expressions for the stress field close to a singular corner (Desmorat, 1996; Desmorat and Leckie, 1998) and we will use the corresponding strain energy in order to approximate the energy density release rate related to a damage analysis. We will use the two scale damage law developed for quasi-brittle materials of Lemaitre (1992).

The case of a bar of steel pulled out of a concrete body is presented.

1.1. Intensity factor and mixed modes

For the crack problem, the intensity factors are defined for each mode, I or II as:

$$\lim_{r \to 0} r^{1-\delta} \sigma_{ij} |_{g=0} = h_1,$$

$$\lim_{r \to 0} r^{1-\delta} \sigma_{ij} |_{g=0} = h_2.$$  (2)

The factors \( h_j \) play a role similar to the stress intensity factors \( k_j \) of interfacial crack problems (Rice, 1988), with the same property of mixed modes \( (h_1 \) and \( h_2 \) are not independent, except for cracks in a single material). For a singularity exponent different from 0.5, it is not possible to relate the \( h \)-factors to an energy release rate as classically done in fracture mechanics. Instead \( h_1 \) and \( h_2 \) are related to the global intensity factor \( h \) for which the dependence on the structure size \( L \) and the applied load \( \sigma_{\infty} \) is given by Reedy (1990):

$$h = \sqrt{h_1^2 + h_2^2} = \tilde{h} \sigma_{\infty} L^{1-\delta},$$  (3)

where \( \tilde{h} \) is a dimensionless shape factor.

1.2. Dundurs parameters

Following (Dundurs, 1967, 1969) for bi-materials composed of elastic isotropic layers A and B with in-plane loading and no applied displacement, the material parameters which define the stress field are reduced to two Dundurs parameters \( \alpha \) and \( \beta \) defined as:

$$\alpha = \frac{\Gamma (1 + \kappa_A) - 1 - \kappa_B}{\Gamma (1 + \kappa_A) + 1 + \kappa_B};$$

$$\beta = \frac{\Gamma (\kappa_A - 1) - \kappa_B + 1}{\Gamma (1 + \kappa_A) + 1 + \kappa_B},$$  (4)

where \( \Gamma \) is the ratio of the shear moduli \( G_B/G_A \) and \( \kappa \) is the plane deformation parameter, \( \kappa = 3 - 4\nu \) for plane strain and \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress state, and \( \nu \) is Poisson’s ratio.

2. Singular field in elasticity

The study applies to a bi-material consisting of quasi-brittle layers. It means that occurrence of plasticity and damage exists at a microscale, but at a mesoscale the behavior remains globally elastic (Lemaitre, 1992). The general singular fields are recalled for anisotropic elasticity (Desmorat, 1996; Desmorat and Leckie, 1998).

For a general anisotropic material the elastic behavior may be written as a linear relationship between the generalized strains \( \varepsilon \) and stresses \( \sigma \):

$$\varepsilon_i = \sum_{j=1}^{6} s_{ij} \sigma_j, \quad \text{or} \quad \sigma_i = \sum_{j=1}^{6} c_{ij} \varepsilon_j,$$  (5)

with the standard notations

$$\{\varepsilon_i\} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}]^T,$$

$$\{\sigma_i\} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T.$$  (6)

with \([\cdot]^T\) being the transpose. The 6 \( \times \) 6 matrices \([s_{ij}]\) and \([c_{ij}]\) are the compliance and stiffness matrices respectively. For two-dimensional problems with in-plane loading orthotropic materials, the strains and stresses reduce to

$$\varepsilon_i = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^T, \quad \{\sigma_i\} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T,$$  (7)

where \([c_{ij}]\) and \([s_{ij}]\) are

$$[c_{ij}] = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{pmatrix},$$

$$[s_{ij}] = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{66} \end{pmatrix}.$$  (8)

For an isotropic material of shear modulus \( G \) and Poisson’s ratio \( \nu \) under plane stress or plane strain

$$s_{11} = s_{22} = (1 + \nu)/8G, \quad s_{12} = (\kappa - 3)/8G$$

and \( s_{66} = 1/G \).

The complex anisotropic representation allows the expression of the displacement \( u \), the resultant
force \( T \) on an arc and the stresses \( \sigma \) as functions of two complex potentials \( f_1 \) and \( f_2 \). This representation is due to Lekhnitskii (1963), Eshelby et al. (1953), Stroh (1958) and has been summarized by Suo (1990). The notations used here are those of Desmorat (1996) and Desmorat and Leckie (1998):

\[
\begin{align*}
\sigma_{ii} &= 2 \Re \left\{ \sum_{j=1}^{2} L_{ij} f_j'(z_j) \right\}, \\
\sigma_{ij} &= -2 \Re \left\{ \sum_{j=1}^{2} L_{ij} f_j(z_j) \right\},
\end{align*}
\]

(9)

where \( f' \) is the derivative of \( f \) and the complex matrices \( A \) and \( L \) are material constants. In order to make the connection between anisotropic and isotropic representations, Stroh (1958) introduced the matrix \( B \) defined as \( (i^2 = -1) \)

\[
B = iAL^{-1}.
\]

(11)

The complex potential method formally satisfies equilibrium, compatibility equations and the elastic stress/strain law, but the specific form of the solution is gained by matching boundary conditions.

For the joint drawn in Fig. 1, two layers \( A \) and \( B \) are bonded together along the interface \( I \) at \( \theta = 0 \). The local geometry of the free edges is represented by the angles \( a \) and \( b \). For such a joint, the singular elastic field is obtained from the complex potential vector \( \mathbf{f} = (f_1, f_2)^T \) of each layer. With the notations defined in the Appendix A:

\[
\begin{align*}
\mathbf{f}_A &= hZ_A^{-1}L_A^{-1} \mathbf{v}, \quad \mathbf{f}_B = hZ_B^{-1}L_B^{-1} \mathbf{w}.
\end{align*}
\]

(12)

The singularity exponent is the smallest positive root of

\[
\text{det } \mathcal{A} (\delta) = 0,
\]

(13)

for the strain energy to remain finite, and where the overbar indicates the conjugate. A singular state of stress will occur only if \( \delta < 1 \). For isotropic layers, the contour of constant \( \delta \) at given angles \( a \) and \( b \) may be drawn in the Dundurs plane. See for instance the review done by Kelly et al. (1992). Fig. 2 is such a plot with \( a = 90^\circ \) and \( b = 180^\circ \). The dependence of the singularity exponent on the angle \( a \) is represented in Table 1 for the case \( (\alpha = -0.76, \beta = -0.30) \) of a bar of steel pulled out of a concrete body (Fig. 3).

The closed-form solution of the vector \( \mathbf{g} \) and the tensor \( \mathbf{F} \) for each layer (Eq. (1)) is derived from

Table 1

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>0</th>
<th>0.94</th>
<th>0.84</th>
<th>0.70</th>
<th>0.66</th>
<th>0.5 + 0.1i</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( 0^\circ )</td>
<td>( 30^\circ )</td>
<td>( 60^\circ )</td>
<td>( 90^\circ )</td>
<td>( 145^\circ )</td>
<td>( 180^\circ )</td>
</tr>
</tbody>
</table>

Fig. 1. Corner consisting of two dissimilar materials.

Fig. 2. Contours of constant \( \delta \) in the Dundurs plane \((a = 90^\circ, b = 180^\circ)\).
Eqs. (9) and (10). Here, we only give the full results for material A, because those for B are found by replacing $v$ by $w$ (for convenience the subscripts A are omitted):

\[
\hat{g} = 2 \text{Re}\left\{A\hat{Z}^d L^{-1} v\right\},
\]

\[
F_{ii} = 2\delta \text{Re}\left\{(L\hat{Z}^{d-1} L^{-1} v)_i\right\},
\]

\[
F_{2i} = -2\delta \text{Re}\left\{(L\mu \hat{Z}^{d-1} L^{-1} v)_i\right\}.
\]

Isotropy is found to be a limiting case of the present analysis (Desmorat and Leckie, 1998). The case $a = b = 180^\circ$ is the interfacial crack problem.

3. Damage analysis

The stress and strain fields close to the corner may be used in order to obtain an order of magnitude of the damaged zone, sometimes called the process zone in fracture mechanics (Knab et al., 1984; Kobayashi et al., 1985), and the size of the crack induced by the singularity. For experimental results on damageable behavior of concrete, see for instance Shah and Maji (1988), Mazars et al. (1988).

The law of damage evolution is derived, for quasi-brittle materials, from the general damage law (Lemaitre, 1992):

\[
\dot{D} = \frac{Y}{S} \hat{\epsilon}, \quad \text{if } p \geq \varepsilon_D,
\]

where $D$ is the isotropic damage variable (the surface density of microcracks or microcavities in any plane)

\[
0 \leq D \leq D_c,
\]

$D_c < 1$ is the critical value of the damage, a constant for each material corresponding to crack initiation; $Y$ is the strain energy density release rate

\[
Y = \frac{\sigma_{eq}^2 R_v}{2E(1 - D)^3},
\]

$R_v$ is the triaxiality function

\[
R_v = \frac{2}{3} (1 + v) + 3(1 - 2v)\left(\frac{\sigma_{H}}{\sigma_{eq}}\right)^2,
\]

$\sigma_{eq}$ is the Von Mises equivalent stress, $\sigma_{H} = \frac{1}{3} \text{tr } \sigma$ is the hydrostatic stress, $E$, $v$ are the Young’s modulus and the Poisson’s ratio; $p$ is the accumulated plastic strain; $S$ is the damage strength characteristic of each material; $\varepsilon_D$ is the damage threshold, the third damage parameter to be identified for each material.

In brittle materials, no plastic strain exists at mesoscale. What is called quasi-brittle is the existence, at a microscale, of weak defects which may induce micro-plasticity and damage. Working on a two scale model of a weak inclusion imbedded in an elastic matrix, it is possible to transform the damage law in terms of the elastic strain of the matrix (Lemaitre, 1992), with the following hypothesis:

- Lin–Taylor hypothesis of equality between the strain $\varepsilon$ at a meso-scale and the strain at a micro-scale $\varepsilon^\mu$,
- inclusion is perfectly plastic having the fatigue limit of the material $\sigma_f$ as yield stress in order to write the yield criterion at a micro-scale as

\[
\frac{\sigma_{eq}^\mu}{1 - D} - \sigma_f = 0,
\]

- elasticity coupled to damage by ($\delta_{ij}$: Kronecker symbol)

\[
\varepsilon_{ij}^\mu = \frac{1 + v}{E} \frac{\sigma_{ij}^\mu}{1 - D} - \frac{3v}{E} \frac{\sigma_{H}^\mu}{1 - D} \delta_{ij}.
\]
\begin{align}
\dot{D} &= \frac{\sigma_f^2}{2ES} R^\mu \dot{\varepsilon}_{eq} \quad \text{if} \quad \sigma_{eq} \geq \sigma_f, \quad \varepsilon_{eq} \geq \varepsilon_D, \tag{24}
\end{align}

with
\[ R^\mu = \frac{2}{3} (1 + v) + 3(1 - 2v) \left( \frac{\sigma_H}{\sigma_f} \right)^2, \]

\[ \dot{\varepsilon}_{eq} = (1 + v)^{-1} \sqrt{\frac{3}{2}} \varepsilon_{ij} \varepsilon_{ij}^{D}, \]

\( \varepsilon_{ij}^D \) being the strain deviator at mesoscale.

A qualitative identification for the one dimensional tension case for which the strain to fracture is \( \varepsilon_f \) allows for a simpler writing. For this case \( \sigma_H = \sigma_f / 3, R^\mu = 1, \dot{\varepsilon}_{eq} = \dot{\varepsilon} \).

Then,
\[ D_c = \frac{\sigma_f^2}{2ES} (\dot{\varepsilon}_R - \dot{\varepsilon}_D) \quad \text{or} \quad \frac{\sigma_f^2}{2ES} = \frac{D_c}{\dot{\varepsilon}_R - \dot{\varepsilon}_D}, \tag{25} \]

from which, for three-dimensional fields the damage rate is
\[ \dot{D} = \frac{D_c}{\dot{\varepsilon}_R - \dot{\varepsilon}_D} R^\mu \dot{\varepsilon}_{eq}, \quad \text{if} \quad \sigma_{eq} \geq \sigma_f, \quad \varepsilon_{eq} \geq \varepsilon_D. \tag{26} \]

Back to the field problem, within a first approximation in which the redistribution of stresses due to the development of damage is not taken into account:

- The damaged zone is defined by
  \[ D \neq 0 \quad \text{or} \quad \varepsilon_{eq} \geq \varepsilon_D. \tag{27} \]

- The cracked zone is defined by the volume (or area in two dimensions), in which the damage has reached its critical value,
  \[ D = D_c. \tag{28} \]

In case of a static loading, this is when
\[ \frac{D_c}{\dot{\varepsilon}_R - \dot{\varepsilon}_D} \int_{\varepsilon_D}^{\varepsilon_{eq}} R^\mu(\sigma_H(\varepsilon_{eq})) \, d\varepsilon_{eq} = D_c \quad \text{or} \]
\[ \int_{\varepsilon_D}^{\varepsilon_{eq}} R^\mu(\sigma_H(\varepsilon_{eq})) \, d\varepsilon_{eq} = \varepsilon_R - \varepsilon_D. \tag{29} \]

In the case of a periodic loading of fatigue, two integrations are needed:

- Integration over one cycle \( N \):
  \[ \frac{\delta D}{\delta N} = \frac{D_c}{\varepsilon_R - \varepsilon_D} \int_{1 \text{ cycle}} R^\mu \, d\varepsilon_{eq}. \tag{30} \]

- Integration over the whole process:
  \[ D = \frac{D_c}{\varepsilon_R - \varepsilon_D} (N - N_0) \int_{1 \text{ cycle}} R^\mu \, d\varepsilon_{eq}, \tag{31} \]

where \( N_0 \) is the number of cycles for \( \varepsilon_{eq} \) to reach \( \varepsilon_D \).

This gives a parametric representation, for \( D = D_c \), of the crack limit as a function of the number of cycles,
\[ \int_{1 \text{ cycle}} R^\mu \, d\varepsilon_{eq} = \frac{\varepsilon_R - \varepsilon_D}{N - N_0}. \tag{32} \]

4. Process and cracked zones

In this part we apply the precedent results to the study of a bar pulled out of a concrete body (Fig. 3). For results on anchor bolt pullout, see for instance Ballarini et al. (1986).

The Young’s modulus and Poisson’s ratio of the bar are \( E_b = 210 \) GPa, \( v_b = 0.3 \) when the material properties of the concrete body considered in tension are \( E = 30 \) GPa, \( v = 0.2, \sigma_f = 1.5 \) MPa, \( S = 2.5 \times 10^{-7} \) MPa, \( \sigma_{eq} = 10^{-4}, \varepsilon_R = 10^{-3} \) (Lemaitre, 1992). The Dundurs parameters for such a joint are \( \alpha = -0.76 \) and \( \beta = -0.30 \). The angle \( b \) is taken equal to \( 180^\circ \) and the geometry is parameterized only by the inclination \( a \). The size of the structure is represented by the bar thickness \( L \).

The information needed to apply the damage analysis in the vicinity of the corner includes expressions for the equivalent stress and strain and for the hydrostatic stress given by the singularity analysis. We consider here the plane strain assumption from which \( F_{33}(0) = v(F_{11} + F_{22}) \) and we will focus on layer A in which failure will occur. From Eqs. (16) and (17), we get
\[ \sigma_{eq} = h \delta^{-1} F_{eq}(\theta), \quad F_{eq} = \sqrt{2} F_{ij} F_{ij}^{D}, \tag{33} \]
\[ \sigma_H = h \delta^{-1} F_{H}(\theta), \quad F_{H} = \frac{1}{3} \text{tr} F, \tag{34} \]
where the singularity exponent depends strongly on the inclination angle.

Because of the elastic behavior at mesoscale, and with the load rate taken into account through, \( h \) (Eq. (3)) we get \( \hat{\sigma}_{eq} = h \hat{r}^{\beta-1} F_{eq} \), then \( \hat{\varepsilon}_{eq} = \hat{\sigma}_{eq} / E \).

For a monotonic loading,

\[
\varepsilon_{eq} = \frac{\sigma_{eq}}{E}. \tag{35}
\]

4.1. Monotonic loading

Let us define the process zone as the domain in which the damage threshold \( \varepsilon_{D} \) at a microscale has been reached for an applied load \( \sigma_{\infty} \) and for the corresponding intensity factor \( h \) (Eq. (3)). In polar coordinates, this zone may be represented by its length \( \ell_{D} \) function of the polar angle \( \theta \). From Eqs. (33) and (35), with \( \varepsilon_{eq} = \varepsilon_{D} \), we get

\[
\ell_{D} = \left( \frac{h F_{eq}(\theta)}{E \hat{r}_{D}} \right)^{1/(1-\beta)}, \tag{36}
\]

represented in Fig. 4.

The cracked zone is defined as the domain in which the failure criterion \( D = D_{c} \) has been reached and is calculated from Eqs. (30) and (35). The cracked length \( \ell_{c} = \ell_{c}(\theta) \), similar to the length of a crack in terms of boundary conditions applied on it (free edges), is then solution of the nonlinear equation:

\[
\varphi(\theta) \left( \frac{h F_{eq}(\theta)}{\sigma_{t}} \right)^{3} \ell_{c}^{2(\beta-1)} + 3 h F_{eq}(\theta) \frac{E \hat{r}_{D}}{\sigma_{t}} \ell_{c}^{\beta-1} = \frac{9G}{\sigma_{t}} (\varepsilon_{R} - \varepsilon_{D}) + \varphi(\theta) \left( \frac{E \hat{r}_{D}}{\sigma_{t}} \right)^{3} 3 \frac{E \hat{r}_{D}}{\sigma_{t}}, \tag{37}
\]

where

\[
\varphi(\theta) = \frac{9(1-2v)}{2(1+v)} \left( \frac{F_{H}(\theta)}{F_{eq}(\theta)} \right)^{2}. \tag{38}
\]

Both process length \( \ell_{D} \) and cracked length \( \ell_{c} \) are proportional to \( h^{1/(1-\beta)} \) (or to \( \sigma_{\infty}^{1/(1-\beta)} \)). But because of the stress triaxiality close to the tip (truncated by the term \( \varphi(\theta) \)), the shape of the process and cracked zones are not identical. The ratio \( \ell_{c} / \ell_{D} \) is smaller than 10% and depends on the polar angle \( \theta \) as represented in Fig. 5 for \( a = 90^\circ \).

In order to minimize the process zone for a given load, a material with a high damage threshold at a microscale should be chosen. At a given load and geometry, the size of the cracked zone will be large for a large ratio \( E(\varepsilon_{R} - \varepsilon_{D})/\sigma_{t} \), or with Eq. (25) for a large ratio \( E^{2}SD_{c}/\sigma_{t}^{2} \); this means for stiff materials, with a high damage strength and with a low fatigue limit.

For the complete determination of the cracked and process length, we need to compute the intensity factor \( h \) as proposed by Stern and Soni (1976) and Leguillon and Sanchez-Palencia (1987). An estimation of the order of magnitude of \( \ell_{c} \) and \( \ell_{D} \) can be obtained by considering roughly the crack problem (crack of length \( 2a \)) for which \( \delta = 0.5 \) and from Eq. (2), \( h(2\pi)^{1/2} = K = \sigma_{\infty}(\pi a)^{1/2} \). The size of the plastic zone may be approximated by

\[
\begin{array}{c}
\frac{l_{c}}{l_{D}} = \frac{91}{10}
\end{array}
\]
\[
\ell_p \approx \frac{K^2}{2\pi \sigma_y} \approx \left( \frac{\sigma_\infty}{\sigma_y} \right)^2 \frac{a}{2},
\]
(39)

where \( \sigma_y = 2.5 \text{ MPa} \), when \( \ell_c \) and \( \ell_D \) are roughly

\[
\ell_D \approx \left( \frac{\sigma_y}{E_{0d}} \right)^2 \approx 0.7\ell_p,
\]
\[
\ell_c \approx \ell_D / 10 \approx \ell_p / 15.
\]
(40)

It means that if the pseudo-plastic zone at a mesoscale is of the order of magnitude of the centimeter, the process zone at a microscale is 0.7 cm large, and the cracked zone, which can be seen here as an increase of the crack length, is of the order of magnitude of the few hundredth of centimeter.

5. Conclusion

The singularity of the elastic stress field close to a sharp notch of any angle in a multimaterial joint having a perfect interface is studied by means of the complex potential method. The exponent of the singularity is obtained as a closed-form solution depending upon the angle of the sharp notch. It is used as an input to an evolution law of damage in order to give the shape and an order of magnitude of the size of the process zone (in which a damage field exists) and of the fully damaged zone (which models the crack induced by the singularity in continuum damage mechanics). The process zone is just smaller than the plastic zone in classical fracture mechanics, when the cracked zone remains very small in comparison.

Appendix A

For the in-plane loading condition, the elastic field may be represented by two holomorphic functions \( f_1(z_1) \) and \( f_2(z_2) \), where the complex coordinates are

\[
z_j = x_j + \mu_j y_j,
\]

introducing the complex numbers \( \mu_j \). For orthotropic materials, \( \mu_j \) are the roots with positive imaginary part of the fourth order equation

\[
\dot{\lambda}^4 + 2\rho \dot{\mu}^{1/2} \mu^2 + 1 = 0.
\]

For plane stress, the constants \( \lambda \) and \( \rho \) which measure the material anisotropy are given by

\[
\lambda = \frac{s_{11}}{s_{22}}, \quad \rho = \frac{1}{2} \left( 2s_{12} + s_{66} \right) \left( s_{11}s_{22} \right)^{-1/2}.
\]

Then

\[
\mu_1 = i\lambda^{-1/4}(n + m), \quad \mu_2 = i\lambda^{-1/4}(n - m) \quad \text{for} \quad 1 < \rho < \infty,
\]
\[
\mu_1 = \lambda^{-1/4}(in + m), \quad \mu_2 = \lambda^{-1/4}(in - m) \quad \text{for} \quad -1 < \rho < 1,
\]
\[
\mu_1 = \mu_2 = i\lambda^{-1/4} \quad \text{for} \quad \rho = 1,
\]

where

\[
n = \left( \frac{1}{3} (1 + \rho) \right)^{1/2}, \quad m = \left| \frac{1}{2} (1 - \rho) \right|^{1/2}.
\]

The positiveness of the strain energy requires that \( \lambda > 0 \) and \(-1 < \rho < \infty\). Plane strain deformation can be treated by a simple change of compliance with

\[
\tilde{s}_{ij} = s_{ij} - \frac{s_{13}s_{13}}{s_{33}}.
\]

Finally the matrix \( A \) and \( L \) defining the stress and displacement fields are

\[
A = \begin{pmatrix}
s_{11}\mu_1^2 + s_{12} & s_{11}\mu_2^2 + s_{12} \\
s_{21}\mu_1 + s_{22}/\mu_1 & s_{21}\mu_2 + s_{22}/\mu_2
\end{pmatrix},
\]
\[
L = \begin{pmatrix}
-\mu_1 & -\mu_2 \\
1 & 1
\end{pmatrix}.
\]

For our singularity problem the complex potentials or materials A and B chosen are of the form

\[
f_k^A(z_k^A) = \phi_k^A(z_k^A)^0,
\]
\[
f_k^B(z_k^B) = \phi_k^B(z_k^B)^0,
\]

where \( \phi_k^A, \phi_k^B (k = 1,2) \) are complex coefficients. Using polar coordinates, the complex coordinates are

\[
z_k^A = r(\cos \theta + \mu_k^A \sin \theta),
\]
\[
z_k^B = r(\cos \theta + \mu_k^B \sin \theta).
\]

Defining for each layer:

\[
\Phi = (\phi_1, \phi_2)^T, \quad Z = \text{diag}(z_1, z_2), \quad f = (f_1, f_2)^T,
\]
the complex potentials may be represented by the vector

\[
f = Z^\dagger \Phi.
\]
It proves convenient to introduce the $2 \times 2$ matrices $X, Y, I, 0$ defined by

$$X = LZ^bL^{-1}, \quad Y = X^{-1}X,$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalue problem due to the writing of the boundary conditions is reduced to

$$A \nu = 0, \quad Y_A \nu + \nu = 0,$$

in which $A$ is the $2 \times 2$ matrix defined by Eq. (13), where the singularity exponent is finally root of

$$\det A(\delta) = 0.$$ 

The stresses and displacements are given by Eq. (1) and Eqs. (15)–(17), with $Z^b = (Z/r)^b, \mu = \text{diag}(\mu_1, \mu_3)$ and where the eigenvectors for each layer $v$ and $w$ are

$$v = \frac{(L \Phi)_A}{h}, \quad w = \frac{(L \Phi)_B}{h}.$$ 

References


