Singularities in bi-materials: parametric study of an isotropic/anisotropic joint

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ABSTRACT. – Problems in fracture mechanics are frequently solved in terms of crack tip singularities. Geometries other than cracks also exhibit singular stresses at points such as corners, edges and interfacial joints. Corners occuring in monolayers or multilayered media have been studied under the assumption that each layer is isotropic. For general elastic plane problems, the present study extends the earlier results to anisotropy. For orthotropic joints, generalized Dundurs parameters are introduced. Isotropic results are a limiting case of the present analysis.

In the vicinity of a singular point, the displacements and stresses may be expressed as a function of the polar coordinates \( r - \theta \) by:

\[
\begin{align*}
\mathbf{u} &= hr^\theta \mathbf{g}(\theta), \\
\mathbf{\sigma} &= hr^{\theta - 1} \mathbf{F}(\theta)
\end{align*}
\]

where \( h \) is the intensity factor and \( \theta \) the singularity exponent \( (0 < \theta < 1) \). The values of \( h \) and \( \theta \) reduce to the stress intensity factor \( K \) and the complex exponent \( 0.5 + i\varepsilon \) for the limiting case of cracks at the interface of dissimilar media.

Using an anisotropic complex potential method, the present analysis gives \( \theta \) as the solution of an eigenvalue problem and as the root of a nonlinear equation \( \det \mathbf{A}(\theta) = 0 \). It leads to a closed-form expression for \( \mathbf{g} \) and \( \mathbf{F} \). The matrix \( \mathbf{A} \) depends on the number of layers at the singular point, their relative elastic properties and the boundary conditions such as free surface or bonded interface close to the singularity corner. A closed-form expression is derived for \( \mathbf{A} \) which depends on 3 generalized Dundurs parameters for a metal matrix composite isotropic metal interface joint. This compares to the 2 Dundurs parameters needed for joints with isotropic layers.

The intensity factor \( h \) of any singularity is determined from a path independent integral, using an extraction function which is more singular than that defining the actual stress state. © Elsevier, Paris.

1. Introduction

Singular solutions are used to solve linear elastic fracture mechanics problems and the procedures are illustrated fully in standard texts such as Liebowitz (1968), Kanninen and Popelar (1985). The singular crack solutions for power-law hardening plasticity were studied by Hutchinson (1968) and Rice and Rosengren (1968). Geometries other than cracks also exhibit singular stresses at corners, edges and interfacial joints. They have been studied for angular corners of isotropic materials by Williams (1952), Bogi (1968), England (1971), Stern and Soni (1976) and Reedy (1990) and for multilayered media by...
Schmauder (1989), Kelly et al. (1992), Reedy (1993). The study has been extended to anisotropic layers by Leguillon and Sanchez Palencia (1987) using a numerical method and then by Desmorat (1996) who derives the closed-form fields. The main results of these studies is that the singular fields for displacement and stress can be expressed as a function of the polar coordinates \( r - \theta \) as (vectors are represented by bold characters and tensors are underlined):

\[
\mathbf{u} = hr^\delta \mathbf{g}(\theta), \quad \sigma = hr^{\delta - 1} F(\theta)
\]

where the singularity exponent \( \delta \) is the solution of an eigenvalue problem defined by the boundary conditions close to the tip. The intensity factor \( h \) of the problem depends on the external load and the geometry of the structure. Obviously for cracks in isotropic materials, \( \delta = 0.5 \) and \( h \) is the stress intensity factor \( K \). For interfacial cracks between dissimilar materials (Rice, 1988), \( \delta = 0.5 + i\varepsilon \) where \( \varepsilon \) is the oscillatory index. For interfacial cracks, \( \varepsilon \) is generally small and has a very localized effect at the crack tip on a scale smaller than the scale of the occurrence of plasticity.

A knowledge of the existence and the strength of a singularity is important in FE computations because it is easy to misjudge a singularity as a stress concentration even with a refined mesh of the zone close to the singular corner. The singular fields also provide a means of obtaining additional information about the stress, strain and displacement fields by using the theoretical singular solution as an input to a very detailed FE computation (Mc Meeking, 1977).

In the present paper, we will focus on joints consisting of anisotropic layers and particularly on the composite joint described in Figure 2 when a bar of metal matrix composite (MMC) is embedded in a uniform isotropic metal. For our application the MMC considered is the Al\(_2\)O\(_3\)/Al composite (materials properties in appendix), with the direction of the fibers coinciding with the longitudinal direction of the bar. The isotropic metallic joint is an aluminium alloy (Young’s modulus \( E_y = 70 \) GPa, Poison’s ratio \( \nu_A = 0.3 \)). The geometry of the joint is a particular case of Figure 1 with an isotropic layer \( A \) and with \( b = 180^\circ \).

The closed-form singular expression of the stress and displacement fields is derived from an anisotropic complex potential method in plane elasticity. Generalized Dundurs parameters are introduced for joints consisting of two orthotropic layers. The exponent \( \delta \) is shown to be the solution of an eigenvalue problem and is the root of a nonlinear equation. The intensity factor \( h \) may be derived from a path independent integral using the reciprocal elastic theorem and a singular function (Stern and Soni, 1976; Babuska and Miller, 1984; Leguillon and Sanchez-Palencia, 1987). The expression for \( h \) will be derived in the present analysis, but its evaluation is left to a later paper.

1.1. INTENSITY FACTOR AND MIXED MODES

As it is for the crack problem, the intensity factors are defined for each mode, 1 or 2 as:

\[
\lim_{r \to 0} r^{1-\delta} \sigma_{\theta}|_{\theta=0} = h_1, \quad \lim_{r \to 0} r^{1-\delta} \sigma_r|_{\theta=0} = h_2
\]
The factors $h_j$ play a role similar to that of the stress intensity factors $k_j$ of interfacial crack problems (Rice, 1988), with the same property of mixed modes as we will see later. The factors $h_1$ and $h_2$ are related to the global intensity factor $h$ for which the dependence on the structure size $L$ and the applied load $\sigma_\infty$ is given by Reedy (1990):

$$h = \sqrt{h_1^2 + h_2^2} = \tilde{h}\sigma_\infty L^{1-\lambda}$$

where $\tilde{h}$ is a dimensionless shape factor. The mixity angle $\psi$ is classically defined as:

$$\tan \psi = \frac{h_2}{h_1}$$

and is not size dependent.

### 1.2. Dundurs Parameters

Following Dundurs (1967, 1969) for bi-materials composed of elastic isotropic layers $A$ and $B$ with in-plane loading and no applied displacement (Fig. 1), the material parameters which define the stress field are reduced to two Dundurs parameters $\alpha_{iso}$, $\beta_{iso}$ defined as:

$$\alpha_{iso} = \frac{\Gamma(1 + \kappa_A) - 1 - \kappa_B}{\Gamma(1 + \kappa_A) + 1 + \kappa_B}, \quad \beta_{iso} = \frac{\Gamma(\kappa_A - 1) - \kappa_B + 1}{\Gamma(1 + \kappa_A) + 1 + \kappa_B}$$

where $\Gamma$ is the ratio of the shear moduli $G_B/G_A$ and $\kappa$ is the plane deformation parameter, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress state, $\nu$ being the Poisson's ratio. With these notations, the value of the oscillatory index $\varepsilon$ is given by:

$$\varepsilon = \frac{1}{2\pi} \ln \left( \frac{1 - \beta_{iso}}{1 + \beta_{iso}} \right)$$

![Fig. 1. - Bi-material consisting of layers A and B: local geometry and Dundurs parameters.](image-url)
For pairs of materials with the same Poisson's ratio, $\kappa_A = \kappa_B = \kappa$, the relationship between the mismatch parameters is simply $\beta_{\text{iso}} = \alpha_{\text{iso}}(\kappa - 1)/(\kappa + 1)$. It is illustrated by Schmauder (1986) and Kelly et al. (1992) that two large classes of joints exist for which either:

\begin{align}
\text{a) } \beta_{\text{iso}} &= 0 \quad \text{or} \quad \text{b) } \beta_{\text{iso}} &= \frac{\alpha_{\text{iso}}}{4}
\end{align}

depending on the material combinations. This means for such circumstances the number of independent material parameters can be reduced to one.

2. The complex potential method for anisotropic elasticity or L.E.S. representation – in-plane loading

The complex potentials for anisotropic bodies have been introduced by Lekhnitskii (1963), Eshelby et al. (1953), Stroh (1958) and summarized by Suo (1990). For convenience, the works of Lekhnitskii, Eshelby and Stroh are referenced as the (L.E.S.) representation. The complex potentials formally satisfy the equilibrium, the compatibility equations and the elastic stress/strain laws but the specific form of the solution is gained by matching boundary conditions.

For a general anisotropic material the elastic behavior may be written as a linear relationship between the generalized strains $\varepsilon$ and stress $\sigma$:

\begin{align}
\varepsilon_i &= \sum_{j=1}^{6} s_{ij} \sigma_j \quad \text{or} \quad \sigma_i &= \sum_{j=1}^{6} c_{ij} \varepsilon_j
\end{align}

with the standard notations,

\begin{align}
\{\varepsilon_i\} &= [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}]^T, \quad \{\sigma_i\} &= [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T
\end{align}

($\cdot)^T$ being the transpose. The $6 \times 6$ matrices $[c_{ij}]$ and $[s_{ij}]$ are the compliance and stiffness matrices respectively.

For two dimensional problems with in-plane loading of orthotropic materials, the strains and stresses reduce to:

\begin{align}
\{\varepsilon_i\} &= [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^T, \quad \{\sigma_i\} &= [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T
\end{align}

where $[c_{ij}]$ and $[s_{ij}]$ are,

\begin{align}
[c_{ij}] &= \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{pmatrix}, \quad [s_{ij}] &= \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{66} \end{pmatrix}
\end{align}

For an isotropic material under plane stress or plane strain deformation, $s_{11} = s_{22} = (1 + \kappa)/8\, G$, $s_{12} = (\kappa - 3)/8\, G$ and $s_{66} = 1/G$. 

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For the in-plane loading condition, the elastic field may be represented by two holomorphic functions $f_1(z_1)$ and $f_2(z_2)$, where,

$$z_j = x + \mu_j y$$

are the complex coordinates (for orthotropic materials these two coordinates are usually different). The complex numbers $\mu_j$ depend on the materials properties. The means of obtaining those numbers have been proposed by Eshelby et al. (1953) and Lekhnitskii (1963) and are summarized by Suo (1990). For orthotropic materials, $\mu_j$ are the roots with positive imaginary part of the fourth order equation,

$$\lambda \mu^4 + 2\rho \lambda^{1/2} \mu^2 + 1 = 0$$

which is valid for both plane stress and plane strain states. For plane stress, the constants $\lambda$ and $\rho$ which measure the material anisotropy are given by:

$$\lambda = \frac{s_{11}}{s_{22}}, \quad \rho = \frac{1}{2} \left( 2s_{12} + s_{66} \right) \left( s_{11}s_{22} \right)^{-1/2}$$

General expressions for the complex numbers are

$$\begin{cases} 
\mu_1 = i\lambda^{-\frac{1}{4}}(n + m), \quad \mu_2 = i\lambda^{-\frac{1}{4}}(n + m) & \text{for } 1 < \rho < \infty \\
\mu_1 = \lambda^{-\frac{1}{4}}(in + m), \quad \mu_2 = \lambda^{-\frac{1}{4}}(in + m) & \text{for } -1 < \rho < 1 \\
\mu_1 = \mu_2 = i\lambda^{-\frac{1}{4}} & \text{for } \rho = 1 
\end{cases}$$

where,

$$n = \left[ \frac{1}{2} \left( 1 + \rho \right) \right]^{1/2}, \quad m = \left[ \frac{1}{2} \left( 1 - \rho \right) \right]^{1/2}$$

The positiveness of the strain energy requires that $\lambda > 0$ and $-1 < \rho < \infty$. For isotropic materials, $\lambda = \rho = 1$, $\mu_1 = \mu_2 = i$. Plane strain deformation can be treated by a simple change of compliance with:

$$s_{ij}' = s_{ij} - \frac{s_{13}s_{j3}}{s_{33}}$$

The value of the different parameters of the FP/A1 layer (see appendix) shown in Figure 2 for a plane strain state are:

$$\lambda = 0.699, \quad \rho = 1.388, \quad \mu_1 = 1.676i, \quad \mu_2 = 0.714i$$

The displacements $u$, stresses $\sigma$ and resultant forces on an arc $T$ (the medium being kept on the left hand side as the observer travel in the positive direction of the arc, Fig. 1) can be derived from the complex potentials and are:

$$u_i = 2\text{Re} \left\{ \sum_{j=1}^{2} A_{ij} f_j(z_j) \right\}, \quad T_i = -2\text{Re} \left\{ \sum_{j=1}^{2} L_{ij} f_j(z_j) \right\}$$
where \( f' \) is the derivative of \( f \). For arbitrary anisotropic materials, general expressions for \( A \) and \( L \) are given by Lekhnitskii (1963). For orthotropic materials, they simplify as:

\[
A = \begin{pmatrix}
\frac{s_{11} \mu_1^2 + s_{12} \mu_2}{\mu_1} & \frac{s_{11} \mu_2^2 + s_{12}}{\mu_2} \\
\frac{s_{21} \mu_1}{\mu_1} & \frac{s_{22} \mu_2^2 + s_{22}}{\mu_2}
\end{pmatrix}, \quad L = \begin{pmatrix}
-\mu_1 & -\mu_2 \\
1 & 1
\end{pmatrix}
\]  

In order to define the stress and displacement field for isotropic materials (Stroh, 1958) introduced the connection between the (L.E.S.) and isotropic complex potentials (Muskhelishvili, 1963) by the means of the positive hermitian matrix

\[
B = iAL^{-1} = \begin{pmatrix}
\frac{2n\lambda^{1/4}(s_{11}s_{22})^{1/2}}{} & i[(s_{11}s_{12})^{1/2} + s_{12}] \\
-i[(s_{11}s_{12})^{1/2} + s_{12}] & \frac{2n\lambda^{1/4}(s_{11}s_{22})^{1/2}}{}
\end{pmatrix}
\]

which comes from the (L.E.S.) representation but remains defined for isotropy. We will later use the \( B \) matrices to derive the generalized Dundurs parameters.

For isotropic materials, \( \rho = 1 \) and the matrices \( A \) and \( L \) of eq. (16)-(17) can no longer be inverted. In this circumstance, another complex representation has to be considered (Suo et al., 1990). The connection between both degenerate and anisotropic stress field is done in paragraph 4.4.

3. Bi-material matrices and generalized Dundurs parameters

For joints consisting of two orthotropic layers \( A \) and \( B \) (Fig. 1), two bi-material matrices \( H \) and \( G \), which are hermitian, positive-definite and involve material elastic constants of both layers are defined as:

\[
H = B_A + B_B, \quad G = B_B + B_A
\]
For a pair of materials with symmetries as high as those defining orthotropy, the expression of the $H$-matrix simplifies and reveals the constant $\beta$ which is the generalization of the second Dundurs parameter $\beta_{iso}$ (Suo, 1990),

\[
\begin{align*}
H &= H_{11} \begin{pmatrix}
1 & -i\beta \left( \frac{H_{22}}{H_{11}} \right)^{1/2} \\
0 & \frac{H_{22}}{H_{11}}
\end{pmatrix} \\
H_{11} &= [2n\lambda^{1/4}(s_{11}s_{22})^{1/2}]_A + [2n\lambda^{1/4}(s_{11}s_{22})^{1/2}]_B \\
H_{22} &= [2n\lambda^{-1/4}(s_{11}s_{22})^{1/2}]_A + [2n\lambda^{-1/4}(s_{11}s_{22})^{1/2}]_B
\end{align*}
\] (24)

\[
\beta = \frac{[s_{11}s_{12}]^{1/2} + s_{22}]_B - [s_{11}s_{12}]^{1/2} + s_{12}]_A}{(H_{11}H_{22})^{1/2}}
\] (25)

For orthotropic materials, the second b-material matrix $G$ becomes (Eq. 23):

\[
G = H_{11} \begin{pmatrix}
\alpha_1 & i\beta \left( \frac{H_{22}}{H_{11}} \right)^{1/2} \\
-i\beta \left( \frac{H_{22}}{H_{11}} \right)^{1/2} & \alpha_2
\end{pmatrix}
\] (26)

where two additional generalized Dundurs parameters $\alpha_1, \alpha_2$ appear. They reduce to $\alpha_{iso}$ when both $A$ and $B$ are isotropic. A fourth parameter $\chi = H_{22}/H_{11}$ also appears in the expressions of $H$ and $G$ which reduces to unity for isotropic layers. For the orthotropic bimaterial, we define

\[
\alpha_1 = \frac{[2n\lambda^{1/4}(s_{11}s_{12})^{1/2}]_B - [2n\lambda^{1/4}(s_{11}s_{12})^{1/2}]_A}{H_{11}}
\] (27)

\[
\alpha_2 = \frac{[2n\lambda^{-1/4}(s_{11}s_{12})^{1/2}]_B - [2n\lambda^{-1/4}(s_{11}s_{12})^{1/2}]_A}{H_{11}}
\] (28)

Generally for orthotropic materials $\lambda^{1/4}$ is close to unity (for the FP/Al $\lambda^{1/4} = 0.915$), and thus the parameters $\alpha_1$ and $\alpha_2$ are almost equal, leading to $\chi \approx 1$, and may be approximated by a mean value $\alpha$. The expression for $\alpha$ is the average value of $\alpha_1$ and $\alpha_2$,

\[
\alpha = \frac{n(\lambda^{1/4} + \lambda^{-1/4})(s_{11}s_{12})^{1/2}]_B - [2n(\lambda^{1/4} + \lambda^{-1/4})(s_{11}s_{12})^{1/2}]_A}{H_{11}}
\] (29)

and using $\lambda^{1/4} + \lambda^{-1/4} \approx 2$ gives the simpler form which be used in the following development:

\[
\alpha = \frac{2n_B(s_{11}s_{12})^{1/2}]_B - 2n_A(s_{11}s_{12})^{1/2]}_A}{H_{11}}
\] (30)
This expression for $\alpha$ and those for $\alpha_1$, $\alpha_2$ and $\beta$ reduce naturally for isotropic layers to the corresponding classical mismatch parameters $\alpha_{\text{iso}}$, $\beta_{\text{iso}}$.

Let us consider now the condition when $A$ is isotropic $B$ is isotropic. Then the matrix $B_A$ and the expression of $H$ and $G$ (Eq. 27 and 26) remain valid, with the equality,

$$[2\pi \lambda^{1/4}(s_{11}s_{12})^{1/2}] = \frac{[2\pi \lambda^{-1/4}(s_{11}s_{12})^{1/2}]}{\lambda} = (1 + \kappa_A)/4G_A$$

The generalized Dundurs parameters for the joint $\alpha_1$, $\alpha_2$ are still given by Eq. (27)-(28) leading to the important simplification which will be used for the parametric study of the composite joint,

$$\chi = \frac{H_{22}}{H_{11}} = 1 + \alpha_2 - \alpha_1$$

In case of two isotropic layers $A$ and $B$ Eq. (32) becomes $\chi = 1$, with $\alpha_1 = \alpha_2 = \alpha = \alpha_{\text{iso}}$.

4. Singular solutions for corners in bi-materials

We now consider the singularities in stress and strain which occur when a corner is formed with two completely anisotropic elastic materials $A$ and $B$ (Fig. 1). Perfect bonding between $A$ and $B$ is assumed.

For general anisotropy, the most general expressions for the matrices $A$ and $B$ have to be considered. For the particular case of orthotropic layers, $A$ and $B$ are defined by Eq. (21). Isotropic results will be obtained as a limiting case of the general anisotropic study (paragraph 4.4).

The singular solutions exhibit a major term close to the singular point $S$ for which the stresses are proportional to $r^{\delta-1}$, and the displacements to $r^{\delta}$. In case of complex $\delta$, the real part of the expressions containing $\delta$ is selected in order to give real fields. The magnitude of the exponent $\delta$ must be positive for the strain energy to remain finite and smaller than 1 for the stress field to be singular. Its value depends on the local geometry (by the angles $\alpha$ and $\beta$) and on the material properties of each layer (by the matrices $B$). Using the L.E.S. representation, the complex potentials or materials $A$ and $B$ are of the form:

$$f_k^A(z_k^A) = \phi_k^A \cdot (z_k^A)^\delta, \quad f_k^B(z_k^B) = \phi_k^B \cdot (z_k^B)^\delta$$

where $\phi_k^A$, $\phi_k^B (k = 1, 2)$ are complex coefficients, introducing four real constants to be determined for each component of the joint. They are related to the normalizing $h$-factor (Eq. 1). Using polar coordinates, the complex coordinates are (Eq. 12):

$$z_k^A = r(\cos \theta + \mu_k^A \sin \theta), \quad z_k^B = r(\cos \theta + \mu_k^B \sin \theta)$$

Defining for each layer:

$$\Phi = [\phi_1, \phi_2]^T, \quad Z = \text{diag}[z_1, z_2], \quad f = [f_1, f_2]^T$$
then the complex potentials may be represented by the vector:

\[ \mathbf{f} = Z^h \Phi \]  

The various field quantities are now from Eq. (16):

\[ \mathbf{u} = \{u_i\} = A \mathbf{f} + \overline{A} \mathbf{f} = A Z^h \Phi + \overline{A} Z^h \overline{\Phi} \]

\[ -\mathbf{T} = \{-T_i\} = L \mathbf{f} + \overline{L} \mathbf{f} = L Z^h \Phi + \overline{L} Z^h \overline{\Phi} \]

4.1. SINGULARITY EXPONENT AS AN EIGENVALUE PROBLEM

It proves convenient to introduce the 2 x 2 matrices \( X, Y, 1, 0 \) defined by

\[ X = L Z^h L^{-1}, \quad Y = \overline{X}^{-1} X, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

The equilibrium and the compatibility equations are automatically satisfied by Eq. (37)-(38). The boundary conditions of the joint of Figure 1 are of two kinds. At the free surfaces where \( \theta = a \) and \( \theta = -b \), \( \mathbf{T} = 0 \) and since the interface \( I \) is assumed fully bonded, \( \mathbf{u}_A = \mathbf{u}_B \) and \( \mathbf{T}_A = \mathbf{T}_B \) for \( \theta = 0 \). The application of these conditions leads to a set of linear equations which can be written:

\[ K(\delta) \mathbf{V} = 0 \]

\( K \) is an 8 x 8 matrix depending on the matrix \( B_A, B_B, Y_A = Y(\theta = a, \delta), Y_B = Y(\theta = b, \delta) \) and \( \mathbf{V} \) depends on 8 real constants to be determined,

\[ K = \begin{pmatrix} Y_A & 1 & 0 & 0 \\ 0 & 0 & Y_B & 1 \\ 1 & 1 & -1 & -1 \\ B_A & -B_A & -B_B & \overline{B}_B \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} (L \Phi)_A \\ (\overline{L} \Phi)_A \\ (L \Phi)_B \\ (\overline{L} \Phi)_B \end{pmatrix} \]

The matrices \( Y \) generally depends on the angle \( a \) (or \(-b\)), the exponent \( \delta \) and on the complex numbers \( \mu_k \) of the considered layer. From Eq. (39),

\[ \begin{cases} Y_A = \overline{L}_A(\overline{Z}_A(a))^{-\delta} \overline{L}_A^{-1} L_A(Z_A(a))^{-\delta} L_A^{-1}, \\ Y_B = \overline{L}_B(\overline{Z}_B(-b))^{-\delta} \overline{L}_B^{-1} L_B(Z_B(-b))^{-\delta} L_B^{-1} \end{cases} \]

The search of nontrivial solution leads to an eigenvalues problem which gives the singularity exponent \( \delta \) as the smallest positive root of \( \det K(\delta) = 0 \). By defining

\[ \mathbf{v} = \frac{(L \Phi)_A}{h}, \quad \mathbf{v} = \frac{(L \Phi)_B}{h} \]
it is possible to reduce the size of the determinant from 8 x 8 to 2 x 2, the eigenvalues problem (40)-(41) becoming

(44) \[ \mathbf{A} \mathbf{v} = 0, \quad Y_A \mathbf{v} + \mathbf{v} = 0 \]

where \( \mathbf{A} \) is the 2 x 2 matrix defined by

(45) \[ \mathbf{A} = \mathbf{B}_A + \mathbf{B}_A Y_A - [\mathbf{B}_B + \mathbf{B}_B Y_B] \cdot (1 - Y_B)^{-1} \cdot (1 - Y_A) \]

and the singularity exponent is now found as the smallest positive root of the 2 x 2 determinant:

(46) \[ \det \mathbf{A}(\delta) = 0 \]

For each layer, \( \mathbf{B} \) and \( Y \) are functions of the stiffness \( s_{11}, s_{22}, s_{12}, s_{66} \). Thus, the singularity exponent \( \delta \) depends on 8 material parameters (for a bi-material) when the geometry is represented by the two angles \( a \) and \( b \). Singular solutions also exist for which \( \delta^* < 0 \) and use shall be one of them later to determine the \( h \)-factor. It should be noticed that if \( \delta \) is a solution then \( \delta^* = -\delta \) is also a solution (Leguillon and Sanchez-Palencia, 1987).

If the composite joint consists of a pair of isotropic materials, or if just one of them is isotropic, Eq. (45)-(46) still define the singular exponent \( \delta \). This will be justified by unifying the writing of the boundary conditions at the interface for both isotropic and anisotropic representations (paragraph 4.4).

4.2. Determination of the Eigenmode

The complex eigenvector \( \mathbf{v} \) solution of Eq. (44) is found by solving the eigenvalue problem:

(47) \[ \mathbf{A} \cdot [1 - (Y_A - 1)^{-1} \cdot (Y_A + 1)] \cdot \text{Re}\{\mathbf{v}\} = 0 \]

where \( \text{Re}\{\mathbf{v}\} \) means the real part of \( \mathbf{v} \). The imaginary part is then given by

(48) \[ \text{Im}\{\mathbf{v}\} = i(Y_A - 1)^{-1} \cdot (Y_A + 1) \cdot \text{Re}\{\mathbf{v}\} \]

We normalize \( \mathbf{v} \) by considering the definition of the intensity factor \( h \) (Eq. 2-3) as well as the expressions of the stresses at the interface (Eq. 20 with \( \theta = 0 \)). This ends up to \( ||\text{Re}\{\mathbf{v}\}||1/2\delta \). The eigenvector \( \mathbf{w} \) is finally found to be:

(49) \[ \mathbf{w} = (1 - Y_B)^{-1} \cdot (1 - Y_A) \mathbf{v} \]

It should also be noticed that the mixity angle is a function of the singularity exponent,

(50) \[ \psi(\delta) = \tan^{-1}\left(\frac{\text{Re}\{v_1\}}{\text{Re}\{v_2\}}\right) \]
4.3. Stresses and Displacements in Anisotropic Layers

The knowledge of the eigenvector $v$ directly leads to the determination of the complex potentials $f$. From Eq. (36) and (43),

$$ f_A = h Z_A^h L_A^{-1} v, \quad f_B = h Z_B^h L_B^{-1} w $$

We get the closed-form expression of the vector $g$ and the tensor $E$ for each layer from Eq. (19)-(20). We only give here the full results for material $B$, because those for $A$ are found by replacing $w$ by $v$ (for convenience the subscripts $B$ are omitted).

$$ g = 2 \text{Re}\{A Z^h L^{-1} w\} $$

$$ F_{1i} = 2 \delta \text{Re}\{(L Z^h L^{-1} w)\}, $$

$$ F_{2i} = -2 \delta \text{Re}\{(L \mu Z^h L^{-1} w)\}, $$

with $Z = \text{diag} [\cos \theta + \mu_1 \sin \theta, \cos \theta + \mu_2 \sin \theta], \mu = \text{diag} [\mu_1, \mu_2]$. For isotropic layers the Muskhelishvili's formulation has to be considered. The related isotropic potentials complex are derived in next paragraph.

4.4. Degenerate Case and Isotropy

For the degenerate case when $\rho = 1$ (which includes isotropy when $\lambda = n = 1$), the complex numbers $\mu_j$ are double roots of Eq. (13) and the complex coordinates reduce to the single value,

$$ z = x + i \lambda^{-1/4} y $$

Furthermore the matrices $A$ and $L$ of Eq. (21) can no longer be inverted, so that the L.E.S. representation is unable to define the various fields. In this circumstance, another complex representation may be used. It was introduced for layers with $\rho = 1$ and is the basis of the rescaling method introduced by Suo et al. (1990a) in order to convert an anisotropic elasticity problem to an equivalent isotropic problem. The resulting complex potential formulation is analogous to Mushkhelishvili’s representation and the holomorphic functions $\omega(z)$ and $\Omega(z)$ define the displacements $u$, stresses $\sigma$ and resultant forces as follow:

$$ \lambda^{1/2} \sigma_{11} + \sigma_{22} = 4 \text{Re}\{\omega'\} $$

$$ \sigma_{22} - \lambda^{1/2} \sigma_{11} + 2 i \lambda^{1/4} \sigma_{12} = 2 [\bar{z} \omega'' + \Omega'] $$

$$ 2 \tilde{G}(u_1 + i \lambda^{1/4} u_2) = \tilde{k} \omega - z \omega' + \bar{\Omega} $$

$$ i (\lambda^{1/4} T_1 + i T_2) = \omega + z \omega' + \bar{\Omega} $$

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where \( (\cdot) \) is the conjugate and the isotropically equivalent shear modulus \( \tilde{G} \) and plane deformation parameter \( \tilde{\kappa} \) are defined for plane stress by:

\[
\tilde{G} = \frac{1}{s_{66}} = \frac{1}{2((s_{11}s_{22})^{1/2} - s_{12})}, \quad \frac{1 + \tilde{\kappa}}{4} = \frac{(s_{11}s_{22})^{1/2}}{(s_{11}s_{22})^{1/2} - s_{12}} = \frac{2(s_{11}s_{22})^{1/2}}{s_{66}}
\]

and their value for plane strain are obtained by the change of compliance of Eq. (14). For isotropic materials we need to find the complex potentials \( \omega(z) \) and \( \Omega(z) \) corresponding to a stress singularity of exponent \( \delta - 1 \). These potentials have the following form,

\[
\left( \begin{array}{c}
\frac{\omega(z)}{\Omega(z)}
\end{array} \right) = \frac{h}{b} z^\delta
\]

Knowing that the L.E.S. representation gives the correct boundary conditions even for isotropic layers, we just have to match those at the interface for both methods, anisotropic and isotropic. Along the interface \( I \), the displacements and resultant forces can be rewritten as \( (x > 0) \):

\[
(62) \quad \mathbf{u} = -ihx^\delta (Bv - \overline{Bv}), \quad \mathbf{T} = -hx^\delta (v = \overline{v}) \quad \text{anisotropic}
\]

\[
(63) \quad \mathbf{u} = hx^\delta (Db + \overline{Db}), \quad \mathbf{T} = hx^\delta (C\mathbf{b} + \overline{C\mathbf{b}}) \quad \text{isotropic}
\]

where:

\[
(64) \quad C = \frac{1}{2} \begin{pmatrix}
    i(\delta - 1) & i \\
   -(1 + \delta) & -1
  \end{pmatrix}, \quad D = \frac{1}{4G} \begin{pmatrix}
    \kappa - \delta & -1 \\
    -i(\kappa + \delta) & -i
  \end{pmatrix}
\]

Matching displacements and forces of both methods gives:

\[
(65) \quad \mathbf{b} = (\overline{D}^{-1}D - \overline{C}^{-1}C)^{-1} \cdot [(\overline{C}^{-1} - i\overline{D}^{-1}R)v + (\overline{C}^{-1} + i\overline{D}^{-1}R)\overline{v}]
\]

Finally, for an isotropic layer, \( g \) and \( F \) are such as:

\[
(66) \quad 2G(g_1 + ig_2) = \kappa b_1 e^{i\delta \theta} - \delta \overline{b_1} e^{-i(\delta - 2)\theta} - \overline{b_2} e^{-i\theta}
\]

\[
(67) \quad F_{11} + F_{22} = 2\delta(b_1 e^{i(\delta - 1)\theta} + \overline{b_1} e^{-i(\delta - 1)\theta})
\]

\[
(68) \quad F_{22} - F_{11} + 2iF_{12} = 2\delta((\delta - 1)(b_1 e^{i(\delta - 3)\theta} + \overline{b_2} e^{i(\delta - 1)\theta})
\]

\[
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\]
Thus, we verify here that for isotropic materials (or degenerate materials if only $\rho = 1$) the elastic solution may be gained by using the anisotropic complex representation in order to derive the eigenvalue problem based on the boundary conditions. By using the degenerate representation (56)-(59), we obtain the stress and displacement fields.

4.5. Remark on the limiting case of a crack

The singular solution at a crack tip between two dissimilar anisotropic elastic materials is obtained here as the limiting case of the corner problem, with $a = -b = \pi$. The eigenvalue problem to be considered is still (44), but the matrix $A$ can be expressed in terms of bi-material matrices: $A = H + e^{2i\theta}H$ as shown by Suo (1990) who completed the stress analysis.

For a crack between orthotropic materials, the oscillatory index $\varepsilon$ has formally the same expression as for the isotropic case (Eq. 6), where $\beta_{\text{iso}}$ is replaced by its orthotropic generalization $\beta$. For a crack between aluminum and FP/Al the value is $\varepsilon = 0.035$.

5. Composite joint

The expressions which have been developed are now applied to the joint geometry shown in Figure 2 which consists of a bar of the MMC FP/Al (material B) embedded in a body of aluminum (material A). The loading is assumed to be in-plane (traction on the bar with an applied stress $\sigma_\infty$) and we will consider a plane strain state. The design parameters are the elastic properties of each layer, the global geometry of the joint, defined by the inclination angle $\alpha$, the embedded length $L$, the thickness of the bar and the total length of the aluminum body.

In the particular case of this joint, $b = \pi$, we show here that the singularity exponent is a function of only the 3 generalized Dundurs parameters $\alpha_1, \alpha_2, \beta$. Furthermore, for most material combinations they may be reduced to a single parameter.

Several different type of failure have been observed (Zok et al., 1995):

(i) Interfacial cracks may initiate at any of the singular point $S, S', S''$, leading to failure by delamination of the joint interface,

(ii) Damage may be induced in the composite B, leading to a crack starting at the corner $S$ and propagating normally to the interface $I$,

(iii) Global failure of the butt joint by delamination, ductile failure at the right hand end of the MMC or by void growth inside the plain metal close to the end.

We will focus on the design of the inclined part for which a failure type (i) or (ii) may occur. The singularity corner $S$ is taken as the origin. The design parameters then reduce to the single value of the angle $\alpha$, since $b$ is now equal to $\pi$. The same study could be done for the other singularity corners $S'$ and $S''$.

5.1. Use of the generalized Dundurs parameters

The singularity exponent for the composite joint is given by Eq. (45)-(46), but the expression of $A$ can be simplified. Because of the free edge at $\theta = -\pi$, $z_1^B$ and $z_2^B$ are both equal to $x = re^{-i\pi}$ and the expression of the matrix $Y_B$ (Eq. 42) becomes,

$$Y_B = Y(\theta = -\pi) = e^{-2i\theta}1$$
For any other angle than \( b = \pi \), the imaginary parts of the complex coordinates \( z_B \) and \( z_B^* \) do not vanish and the two parameters \( \mu_1^B \) and \( \mu_2^B \) have to be taken into account. The material \( A \) being isotropic, \( Y_\alpha \) is defined as the limit of Eq. (42) for \( \mu_k \rightarrow i \) and only depends on the angle \( \alpha \) and the exponent \( \delta \),

\[
Y_\alpha = e^{2i\delta \alpha} \begin{pmatrix}
1 - 2s^2\delta^2 + 2i\delta sc \\
2i\delta(1 + \delta)s^2 \\
1 - 2s^2\delta^2 - 2i\delta sc
\end{pmatrix}
\]

where \( c = \cos \alpha \), \( s = \sin \alpha \). For the composite joint, with \( b = \pi \), the matrix \( A \) may be now expressed in terms of bi-materials matrices \( (H, G) \) and reduces to

\[
A = H + e^{-2i\delta \pi} H - \frac{Y_\alpha - e^{2i\delta \pi} 1}{1 - e^{2i\delta \pi}}
\]

The matrix \( B_\alpha \) as well as the expressions of \( H \) and \( G \) (Eq. 24 and 26) remain valid for an isotropic material \( A \). Equations (24), (26), (32) show that the matrix \( A/H_{11} \) is a function of the exponent \( \delta \), of the angle \( \alpha \) and of the 3 generalized Dundurs parameters. Thus, from \( \det A = 0 \), the singularity exponent \( \delta \) only on or, \( (\alpha_1, \alpha_2, \beta, \alpha) \).

5.2. Parametric Study of the Singularity Exponent

For orthotropic MMC's, \( \lambda^{1/4} \) is generally close to unity, so that the parameters \( \alpha_1 \) and \( \alpha_2 \) are almost equal to \( \alpha \). It means that the singularity exponent depends roughly on the two parameters \( \alpha \) and \( \beta \) (Eq. 25 and 30). For the Al-FP/A1 joint (of the category for which \( \beta \approx \alpha/4 \)),

\[
\alpha_1 = -0.442, \quad \alpha_2 = -0.387, \quad \alpha = -0.416, \quad \beta = -0.109
\]

The fact that layer \( B \) is a bar (and that \( z_B z_B^* = z \) for \( \theta = -\pi \)) has played a major role in the simplification in the number of independent parameters of the eigenvalue problem. For the case of two orthotropic materials bonded together with the local geometry generally defined by the angles \( a \) and \( b \), the singularity exponent was a function of 8 independent material parameters (6 if layer \( A \) is isotropic). For the composite joint with \( A \) isotropic and \( b = 180^9 \), \( \beta \) is a function only of 3 generalized Dundurs parameters, and in most cases only of \( \alpha \) and \( \beta \) or if we assume the relation \( \beta = \alpha/4 \) of the single \( \alpha \) (Fig. 3). This reduction is of course helpful for designing the joint and makes the parametric study quite simple.
Singularities in Bi-Materials

Figure 4 illustrates the various of the singularity exponent for the composite joint as a function of the angle \( \alpha \) for the exact value of the generalized Dundurs parameters and for their approximate value \( \alpha_1 \approx \alpha_2 \approx \alpha \). Another way to represent the result is to plot the isovalue of \( \delta \) in the \( \alpha-\beta \) plane for different angles \( \alpha \) (Fig. 3, \( \alpha = 90^\circ \)). For any positive \( \alpha \) or any positive \( \beta \) it can be seen that the corner almost behaves as a crack (\( \delta \approx 0.5 \)). The \( \alpha \)-approximation makes it possible to use pre-existing results of isotropic layers joints (see for instance the review done by Kelly (1992) in the \( \alpha_{iso}-\beta_{iso} \) plane). The correct value of the singularity exponent for joints with \( \beta = \pi \) will be found by replacing \( \alpha_{iso} \) by \( \alpha \) and \( \beta_{iso} \) by \( \beta \).

By considering the general properties \( \beta \approx 0 \), \( \beta \approx \alpha/4 \) valid for most of the joints of dissimilar materials, we can plot the curve Singularity Exponent versus Angle \( \alpha \) for
different values of the parameter $\alpha = [-0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75]$ (Fig. 5). It shows that the singularity exponent depends mostly (for a given geometry) on the single material parameter $\alpha$. For angles smaller than 90°, the value of $\delta$ decreases almost linearly and for angles larger than 90°, the singular behavior is almost the same as that for a crack $\delta \approx 0.5$.

![Graph showing exponent $\delta$ vs Angle $\alpha$ for different values of $\beta$.](image)

**Fig. 5.** Exponent $\delta$ vs Angle $\alpha$ curves for different values of $\alpha$ and $\beta$ (with $h = 180°$).

6. Determination of $h$: the $\psi$-integral

Considering two solutions of an elasticity problem, displacements $u^a$, $u^b$, associated stresses $\sigma^a$, $\sigma^b$, satisfying the same boundary conditions but obtained by different methods, one of which may come from a global finite element solution and the other from the study of the singular solution, the difference of stored energy $W^a - W^b$ in the body $\Omega$ (Fig. 1) due to these different solutions may be expressed as a path independent integral form:

$$W^a - W^b = \psi(u^a, u^b) = \frac{1}{2} \int_C (\sigma^a \cdot n \cdot u^b - \sigma^b \cdot n \cdot u^a) dS$$
where the stored energy $W$ in the whole body each solution is:

\begin{equation}
W^k = \frac{1}{2} \int_{\Omega} \sigma^k : \varepsilon^k \, dV
\end{equation}

and $C$ is any contour around the corner $O$, inside which no external load is applied.

The path independence of $\psi$ was demonstrated by Stern (1976) and was used to determine the stress intensity at fixed-free corners. Leguillon (1993), Leguillon and Sanchez-Palencia (1987, 1991) extended this method to any geometry including 2- or 3-D singularities. The main idea is to compare the singular solution $u = h r^\alpha g$, $\sigma = h r^{\alpha-1} F$ to a reference solution (displacements $U$, associated stresses $\Sigma$ coming for example from a FE computation), by the introduction of the extraction solution $u^* = r^{\alpha^*} g^*$, $\sigma^* = r^{\alpha^*-1} F^*$, $\delta^* = -\delta$ (Babuska and Miller, 1984). This last solution is more singular than that which describes the state in the vicinity of the corner and has the property to make the integral $\psi_{C_1}(u, u^*)$ finite. The solution $u^*, \sigma^*$ is fully determined by the foregoing analysis from the knowledge of the singularity exponent $\delta^*$ (eigenproblem 45-46). Using the path independence, the comparison between $u$ and $U$ may be written

\begin{equation}
\psi_{C_1}(u, u^*) = \psi_{C_2}(U, u^*)
\end{equation}

which leads to:

\begin{equation}
h = \frac{\psi_{C_2}(U, u^*)}{\psi_{C_1}(r^\alpha g, u^*)}
\end{equation}

The two contours $C_1$ and $C_2$ may be any contour of course but it seems judicious to take for $C_1$ a circle of center $O$ and of radius going to zero. The integral $\psi_{C_1}(r^\alpha g, u^*)$ can be accurately calculated. It has been shown that even a crude reference solution for $U$ can still give a good estimation of the intensity factor (Stern and Soni, 1976). It is also recalled that the scaling law (Eq. 3) can be used to extend the knowledge of the intensity factor for a given sample to any specimen having the same shape but a different size: in that case the shape factor $\tilde{h}$ remains constant.

**Conclusion**

The present analysis allows fast calculations of 2D-elastic singular strains, stresses and displacements close to corners, edges or interface ends of joints constituted of anisotropic layers. The exact expression of the singular fields is given. isotropic results are derived as a limiting case of the general anisotropic study.

The singularity exponent of the MMC joint studied here depends on the local geometry defined by the angle $\alpha$ and on the materials properties defined by the three generalized Dundurs parameters, one of which was known previously and two have been introduced in the present study. They make the design of the joint quick and the use of previous knowledge on joints constituted of isotropic layers can be extended to the MMC joint. Figure 3 shows the plot of constant singularity exponent $\delta$ in the $\alpha-\beta$ plane (eqs. 25
and 30) while Figure 5 represents the variation of $\delta$ with respect to the local geometry (defined by the angle $\alpha$) for different bi-materials (represented by the single mismatch parameter $\alpha$ under the assumptions $\beta \approx 0$ and $\beta = \alpha/4$). For angles smaller then $90^\circ$, the value of $\delta$ decreases almost linearly and for angles larger then $90^\circ$, the singular behavior is almost the same as that for a crack $\delta \approx 0.5$).

The intensity factor $h$ is determined from a path independent integral valid not only for cracks but for any other geometry. We may need FE computations to find $h$, but not necessarily: analytical solutions giving an idea of a high stress concentration may be sufficient.

An infinite stress is not really physical: at high level of stress, plasticity and damage are induced, limiting or erasing the singularity. The present analysis does not give directly the final stress state but may be a useful tool for computing the plastic or damaged zones (Desmorat, 1996; Desmorat and Lemaitre, 1996):

- An estimation of the plastic zone shape and size can easily found by introducing the yield stress.
- Neglecting the stress redistribution, it is possible to make local post-calculations of the nonlinear evolution laws of the material. The final stress state will take into account plastic and damage. These calculations do not need any FE computations.
- One can use the linear singular solution as an input to a very detailed FE computation: in a zone localized around the singular point, the nonlinear behavior is taken into account and the singular stress field is used as boundary condition.
- Fatigue behavior may be also studied by assuming that the stress field under cyclic loading may be derived from the present analysis.

Appendix

For a transversely isotropic composite, such as FP/Al system, the stiffness matrix $\mathbf{S}$ introduces the five longitudinal ($L$) and transverse ($T$) elastic properties $E_L$, $E_T$, $G_L$, $G_T$, $\nu_L$, $\nu_T$ the longitudinal direction being parallel to the fibers.

\[
S = \begin{pmatrix}
\frac{1}{E_L} & \frac{\nu_L}{E_L} & \frac{\nu_L}{E_L} & 0 & 0 & 0 \\
\frac{\nu_L}{E_L} & \frac{1}{E_T} & \frac{-\nu_T}{E_T} & 0 & 0 & 0 \\
\frac{\nu_L}{E_L} & \frac{-\nu_T}{E_T} & \frac{1}{E_T} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_T} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_L} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_L}
\end{pmatrix}
\]

\[G_T = \frac{E_T}{2(1+\nu-T)}\]
where these elastic properties have been determined experimentally by Jansson (1991):

\( E_L = 225 \text{ GPa}, \quad G_L = 58 \text{ GPa}, \quad \nu_L = 0.28 \)  
\( E_T = 150 \text{ GPa}, \quad G_T = 55 \text{ GPa}, \quad \nu_T = 0.31 \)

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